INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

1 Find: **(a)** $\int \frac{dx}{\sqrt{9-x^2}}$ **(b)** $\int \frac{dx}{9+x^2}$ **(c)** $\int \frac{-1}{\sqrt{16-x^2}}dx$

a) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$

b) $\int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2} = \frac{1}{3}\int \frac{3}{3^2+x^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

c) $\int \frac{-1}{\sqrt{16-x^2}}\,dx = \int \frac{-1}{\sqrt{4^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{4}\right) + C$

3 $\int \frac{x^2+2}{x^2+1}dx = \ldots$

A $x+\ln(x^2+1)+C$ $\boxed{\textbf{B}}$ $x+\tan^{-1}x+C$ **C** $x-\ln(x^2+1)+C$ **D** $x-\tan^{-1}x+C$

$\int \frac{x^2+2}{x^2+1}\,dx = \int \frac{x^2+1\ +1}{x^2+1}\,dx = \int \left[1 + \frac{1}{x^2+1}\right] dx$

$= x + \int \frac{1}{x^2+1}\,dx = x + \tan^{-1}x + C$

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4 Find: **(a)** $\displaystyle\int \frac{dx}{x^2+4x+5}$ **(b)** $\displaystyle\int \frac{dx}{x^2-6x+10}$ **(c)** $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$

a) $x^2+4x+5 = (x+2)^2 - 4 + 5 = (x+2)^2 + 1$

So $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{1+(x+2)^2}$

We do a change of variable $u = x+2$ so $\dfrac{du}{dx} = 1 \Leftrightarrow du = dx$

—— $\displaystyle = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(x+2) + C$

b) $x^2-6x+10 = (x-3)^2 - 9 + 10 = (x-3)^2 + 1$

So $\displaystyle\int \frac{dx}{x^2-6x+10} = \int \frac{dx}{(x-3)^2+1}$ Say $u = x-3$, $du = dx$

$\displaystyle\int \frac{dx}{x^2-6x+10} = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1}(x-3) + C$

c) $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$

$3+2x-x^2 = -x^2+2x+3$

—— $= -(x^2-2x)+3$

—— $= -[(x-1)^2-1]+3$

—— $= -(x-1)^2+1+3$

—— $= 4-(x-1)^2$

$\displaystyle = \int \frac{dx}{\sqrt{2^2-(x-1)^2}}$

Let $u = x-1$ $du = dx$

$\displaystyle = \int \frac{du}{\sqrt{2^2-u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + C = \sin^{-1}\left(\frac{x-1}{2}\right) + C$

4 Find: (g) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ (h) $\int \frac{x+1}{\sqrt{4-x^2}}\,dx$ (i) $\int \frac{dx}{x^2+4x+6}$

g) $x = \sin u$ so $\frac{dx}{du} = \cos u$ $dx = \cos u \times du$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2 u}{\sqrt{1-\sin^2 u}} \times \cos u\,du = \int \frac{\sin^2 u}{\cos u} \times \cos u\,du$$

$$\text{---} = \int \sin^2 u\,du = \int \left[\frac{1-\cos 2u}{2}\right] du = \frac{1}{2}\int du - \frac{1}{2}\int \cos 2u\,du$$

$$\text{---} = \frac{u}{2} - \frac{1}{2}\,\frac{\sin 2u}{2} + C = \frac{u}{2} - \frac{\sin 2u}{4} + C$$

$$\sin 2u = 2\sin u \cos u = 2x\sqrt{1-\sin^2 u} = 2x\sqrt{1-x^2}$$

So $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\sin^{-1}x}{2} - \frac{2x\sqrt{1-x^2}}{4} + C$$

or $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$$

h) $$\int \frac{x+1}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{dx}{\sqrt{4-x^2}}$$

First, we deal with $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C$ (easy)

Second: $\int \frac{x}{\sqrt{4-x^2}}\,dx$. $u = 4-x^2$ so $\frac{du}{dx} = -2x$

$du = -2x\,dx$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \frac{(-2x)\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$\text{---} = -\frac{1}{2}\,\frac{u^{1/2}}{1/2} = -u^{1/2} = -\sqrt{u} = -\sqrt{4-x^2}$$

So $$\int \frac{x+1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C$$

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i) $\displaystyle\int \frac{dx}{x^2+4x+6} = \int \frac{dx}{(x+2)^2-4+6} = \int \frac{dx}{(x+2)^2+2}$

Now, as $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + C$

So $\displaystyle\int \frac{dx}{x^2+4x+6} = \int \frac{dx \times (x+2)'}{(\sqrt{2})^2+(x+2)^2}$

$\displaystyle —— = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right) + C$

5 Evaluate: (a) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}$ (b) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ (c) $\int_0^1 x\sqrt{1-x^2}\,dx$

a) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2} = \int_0^{\sqrt{3}} \frac{dx}{3^2+x^2} = \frac{1}{3}\int_0^{\sqrt{3}} \frac{3}{3^2+x^2}\,dx = \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^{\sqrt{3}}$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \tan^{-1}0\right] = \frac{1}{3}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 0\right]$

$= \frac{1}{3} \times \frac{\pi}{6} = \pi/18$

b) $\int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}} = \int_0^{1/4} \frac{dx}{2\sqrt{\left(\frac{1}{2}\right)^2 - x^2}} = \frac{1}{2}\int_0^{1/4} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{1/2}\right)\right]_0^{1/4} = \frac{1}{2}\left[\sin^{-1}(2x)\right]_0^{1/4} = \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}0\right]$

$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\,\frac{\pi}{6} = \frac{\pi}{12}$

c) $\int_0^1 x\sqrt{1-x^2}\,dx$ $\quad u = 1-x^2 \quad \frac{du}{dx} = -2x$

so $du = -2x\,dx$

$= \int_1^0 -\frac{du}{2}\sqrt{u} = \frac{1}{2}\int_0^1 \sqrt{u}\,du$ $\quad$ For $x=0$ $u=1$; For $x=1$ $u=0$

$= \frac{1}{2}\left[\frac{u^{1/2+1}}{1/2+1}\right]_0^1 = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{1}{3}\left[u^{3/2}\right]_0^1 = \frac{1}{3}$

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5 Evaluate: (h) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ (i) $\displaystyle\int_0^{4} \frac{dx}{x^2-2x+4}$

h) Change of variable
$x = \cos\theta$ $\quad \frac{dx}{d\theta} = -\sin\theta$ so $dx = -\sin\theta \, d\theta$

When $x = 0$ $\quad \theta = \pi/2$
When $x = 1/2$ $\quad \theta = \pi/3$

So $\displaystyle\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}} = \int_{\pi/2}^{\pi/3} \frac{-\sin\theta \, d\theta}{(\sin^2\theta)^{3/2}}$

$\displaystyle — = \int_{\pi/3}^{\pi/2} \frac{1}{\sin^2\theta} d\theta = \Big[-\cot\text{an}\,\theta\Big]_{\pi/3}^{\pi/2} = \Big[\cot\text{an}\,\theta\Big]_{\pi/2}^{\pi/3}$

So $\displaystyle\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}} = \cot\text{an}\,\frac{\pi}{3} - \cot\text{an}(\pi/2) = \frac{\cos(\pi/3)}{\sin(\pi/3)} - \frac{\cos(\pi/2)}{\sin(\pi/2)} = \frac{1}{\sqrt{3}}$

i) $x^2 - 2x + 4 = (x-1)^2 + 3$ So

$\displaystyle\int_0^4 \frac{dx}{x^2-2x+4} = \int_0^4 \frac{dx}{(x-1)^2+3}$ Change of variable $X = x - 1$
$dX = dx$

when $x = 0$, $X = -1$; when $x = 4$, $X = 3$

$\displaystyle\int_0^4 \frac{dx}{x^2-2x+4} = \int_{-1}^{3} \frac{dX}{X^2+3} = \frac{1}{\sqrt{3}} \int_{-1}^{3} \frac{\sqrt{3}}{X^2+(\sqrt{3})^2} dX$

$\displaystyle — = \frac{1}{\sqrt{3}} \left[\tan^{-1}\left(\frac{X}{\sqrt{3}}\right)\right]_{-1}^{3} = \frac{1}{\sqrt{3}} \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$

$\displaystyle — = \frac{1}{\sqrt{3}} \left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{\sqrt{3}} \left[\frac{1}{3} + \frac{1}{6}\right] = \frac{\pi}{\sqrt{3}} \times \left(\frac{2+1}{6}\right)$

So $\displaystyle\int_0^4 \frac{dx}{x^2-2x+4} = \frac{\pi\sqrt{3}}{6}$

5 **Evaluate:** (m) $\int_{-\frac{7}{4}}^{\frac{3}{4}} \frac{dx}{\sqrt{6-x-x^2}}$ (n) $\int_0^a \sqrt{a^2-x^2}\,dx$ (o) $\int_{\tan x}^{\cot x} \frac{dt}{1+t^2}$, $0 < x < \frac{\pi}{2}$

m) $6 - x - x^2 = -[x^2+x-6] = -\left[\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - 6\right]$

So $6-x-x^2 = -\left[\left(x+\frac{1}{2}\right)^2 - \frac{25}{4}\right] = \frac{25}{4} - \left(x+\frac{1}{2}\right)^2$

$\text{———} = \left(\frac{5}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2$

$$\int_{-7/4}^{3/4} \frac{dx}{\sqrt{6-x-x^2}} = \int_{-7/4}^{3/4} \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}}$$

Now, as: $\displaystyle\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + C$

So $\displaystyle\int_{-7/4}^{3/4} \frac{dx}{\sqrt{6-x-x^2}} = \left[\sin^{-1}\left(\frac{x+1/2}{5/2}\right)\right]_{-7/4}^{3/4} = \left[\sin^{-1}\left(\frac{2x+1}{5}\right)\right]_{-7/4}^{3/4}$

$\text{———} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(\frac{-\pi}{6}\right) = \frac{\pi}{3}$

n) $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$

$\sqrt{a^2-x^2}$ suggests the substitution $x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$ so $dx = a\cos\theta\,d\theta$

When $x = 0$ $\theta = 0$

When $x = a$ $\sin\theta = 1$ so $\theta = \pi/2$

n) (cont.) $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta} \times a\cos\theta\, d\theta$

$$—— = \int_0^{\pi/2} a\cos\theta \times a\cos\theta\, d\theta = a^2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$ so $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$—— = a^2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\left[\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta\right]$$

$$—— = \frac{a^2}{2}\left[\frac{\pi}{2} + \left[\sin 2\theta\right]_0^{\pi/2}\right] = \frac{a^2}{2}\left[\frac{\pi}{2} + \left(\sin\pi - \sin 0\right)\right] = \frac{a^2\pi}{4}$$

o) $\displaystyle\int_{\tan x}^{\cot x} \frac{dt}{1+t^2}$

As $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + C$, so:

$$\int_{\tan x}^{\cot x} \frac{dt}{1+t^2} = \left[\frac{1}{1}\tan^{-1}\left(\frac{t}{1}\right)\right]_{\tan x}^{\cotan x} = \left[\tan^{-1}(t)\right]_{\tan x}^{\cotan x}$$

$$—— = \tan^{-1}(\cotan x) - \tan^{-1}(\tan x)$$

But $\cotan x = \dfrac{\cos x}{\sin x} = \dfrac{\sin\left(\pi/2 - x\right)}{\cos\left(\pi/2 - x\right)} = \tan\left(\dfrac{\pi}{2} - x\right)$, so

$$\int_{\tan x}^{\cotan x} \frac{dt}{1+t^2} = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - x\right)\right) - x = \frac{\pi}{2} - x - x$$

$$\int_{\tan x}^{\cotan x} \frac{dt}{1+t^2} = \frac{\pi}{2} - 2x$$