THE SECOND DERIVATIVE AND TURNING POINTS

1 For $y = 2x^3 + 3x^2 - 12x + 2$, find any stationary points and determine their nature. Sketch the curve, showing the turning points and any points of inflection.

$f(x) = 2x^3 + 3x^2 - 12x + 2$ so $f'(x) = 6x^2 + 6x - 12$

$f'(x) = 6\left[x^2 + x - 2\right]$

$\Delta = 1 - 4 \times (-2) = 9 = 3^2$

two roots $x_1 = \dfrac{-1-3}{2} = -2$ and $x_2 = \dfrac{-1+3}{2} = 1$

$f'(x) = 6(x+2)(x-1)$

$f'(x) = 0$ for $x = -2$ and $x = 1$

$f''(x) = 12x + 6$ so $f''(x) = 0$ for $x = -\frac{1}{2}$

For $x = -1/2$ $f(x) = 2 \times \left(\frac{-1}{2}\right)^3 + 3\left(\frac{-1}{2}\right)^2 - 12 \times \left(\frac{-1}{2}\right) + 2 = 8.5$

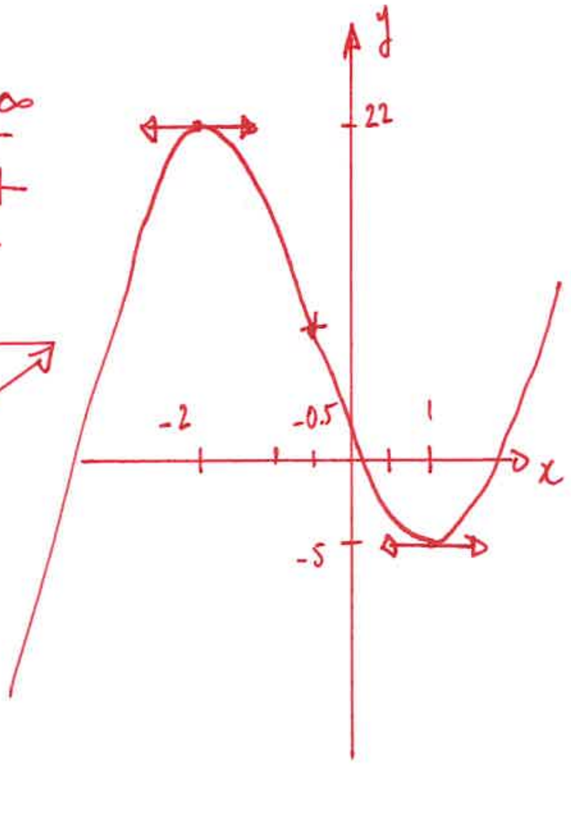
For $x = 1$ $f(x) = 2 \times 1^3 + 3 \times 1^2 - 12 \times 1 + 2 = -5$

For $x = -2$ $f(x) = 2 \times (-2)^3 + 3 \times (-2)^2 - 12 \times (-2) + 2 = 22$

So turning points are $(-2, 22)$ and $(1, -5)$. Inflection at $(-0.5, 8.5)$

$\lim_{x \to +\infty} f(x) = +\infty$ $\lim_{x \to -\infty} f(x) = -\infty$

x	$-\infty$		-2		-0.5		1		$+\infty$
$f'(x)$		$+$	0		$-$		0	$+$	
$f''(x)$					0				
Shape		$\nearrow$	22	$\searrow$	8.5	$\searrow$	-5	$\nearrow$	



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3 Find the local maxima, minima and points of inflection of $f(x) = x^2(3 - x)$. Sketch the graph of f.

$f(x) = 3x^2 - x^3 = -x^3 + 3x^2$

$f'(x) = -3x^2 + 6x = 3x(-x+2)$ so $f'(x) = 0$ for $x = 0$ and $x = 2$

$f''(x) = -6x + 6 = 6(-x+1)$ so $f''(x) = 0$ for $x = 1$

For $x = 0$, $f(x) = 0$ so $(0,0)$ ~~inflection point~~

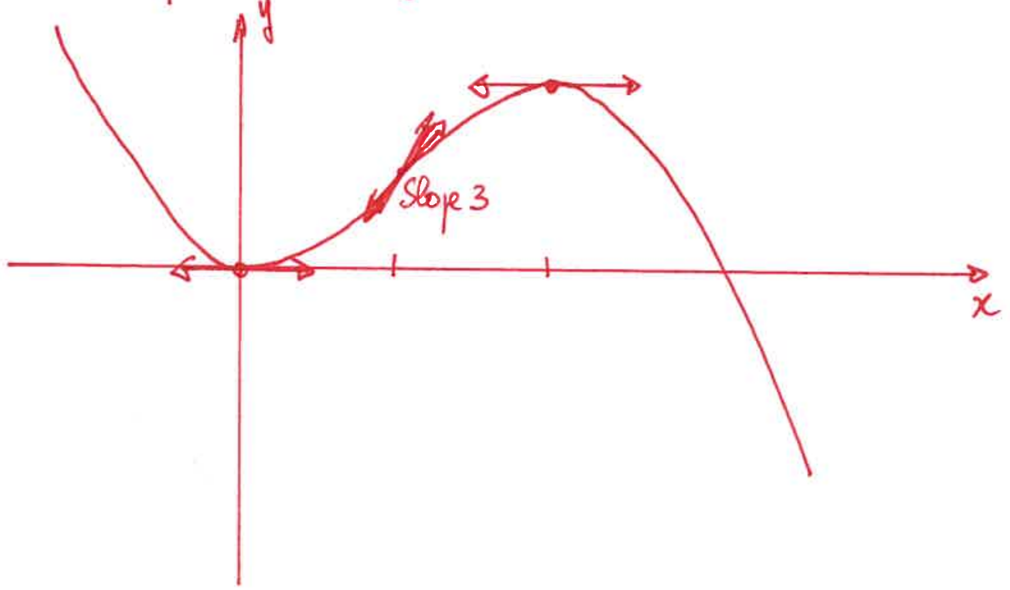
For $x = 1$, $f(x) = 3 \times 1^2 - 1^3 = 3 - 1 = 2$ so $(1,2)$

For $x = 2$, $f(x) = 3 \times 2^2 - 2^3 = 12 - 8 = 4$ so $(2,4)$

Further $\lim\limits_{x \to +\infty} f(x) = -\infty$ and $\lim\limits_{x \to -\infty} f(x) = +\infty$

$f'(1) = 3(-1+2) = 3$

x	$-\infty$		0		1		2		$+\infty$
$f'(x)$		$-$	0		3		0	$-$	
$f''(x)$					0				
Shape	$+\infty$	$\searrow$	0	$\nearrow$	2	$\nearrow$	4	$\searrow$	$-\infty$



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5 A function $f(x)$ is defined by $y = x^3(x-2)$. $= x^4 - 2x^3$

(a) Find the coordinates of the turning points of $y = f(x)$.

(b) Find the coordinates of the points of inflection.

(c) Hence sketch the graph of $y = f(x)$, showing the turning point, the points of inflection and the points where the curve meets the x-axis.

(d) What is the minimum value of $f(x)$ for $-1 \le x \le 3$?

a) $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$

So $f'(x) = 0$ for $x = 0$ and $x = \frac{3}{2}$

At $x = 0$ $f(x) = 0$ At $x = \frac{3}{2}$ $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3\left(\frac{3}{2} - 2\right) = -\frac{27}{16} \sim -1.7$

So $(0,0)$ and $\left(\frac{3}{2}, \frac{-27}{16}\right)$

b) $f''(x) = 12x^2 - 12x = 12x(x-1)$

So $f''(x) = 0$ for $x = 0$ and $x = 1$

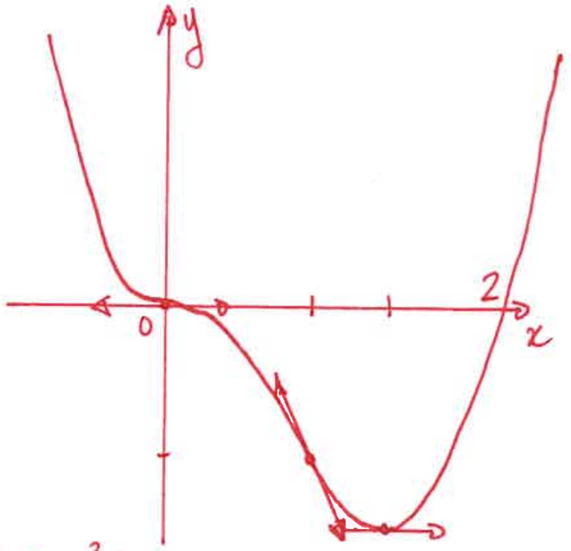
At $x = 0$ $f(0) = 0$ and at $x = 1$ $f(1) = 1^3(1-2) = -1$

So 2 points $(0,0)$ and $(1,-1)$ for which $f''(x) = 0$

at $x = 1$ $f'(1) = 2 \times 1^2(2 \times 1 - 3) = -2$

c) The graph meets the x-axis at $x = 0$ and $x = 2$

x	$+\infty$		0	1	$\frac{3}{2}$		$+\infty$
$f'(x)$		$-$	0	-2	0	$+$	
$f''(x)$			0	0			
Shape	$+\infty$		0	-1	$-\frac{27}{16}$		



d) The minimum for $-1 \leqslant x \leqslant 3$ is at $x = 3/2$

and it's $-\frac{27}{16}$

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8 Let $f(x) = x^4 - x^2$. $= x^2(x^2-1) = x^2(x-1)(x+1)$

(a) Find the coordinates of the points where the curve crosses the axes.
(b) Find the coordinates of the stationary points and determine their nature.
(c) Find the coordinates of the points of inflection.
(d) Sketch the graph of $y = f(x)$ for $-1.5 \le x \le 1.5$, indicating clearly the intercepts, stationary points and points of inflection.
(e) For what values of x is the curve concave down?

a) $f(x)$ crosses the x-axis at $x=-1$, $x=0$ and $x=1$

b) $f'(x) = 4x^3 - 2x = 2x(2x^2-1)$

So $f'(x)=0$ for $x=0$ $x = \frac{-1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}} \sim 0.7$

At $x = \frac{-1}{\sqrt{2}}$ $f\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$ at $x = \frac{1}{\sqrt{2}}$ $f\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$ too

$\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$

c) $f''(x) = 12x^2 - 2 = 2(6x^2-1)$

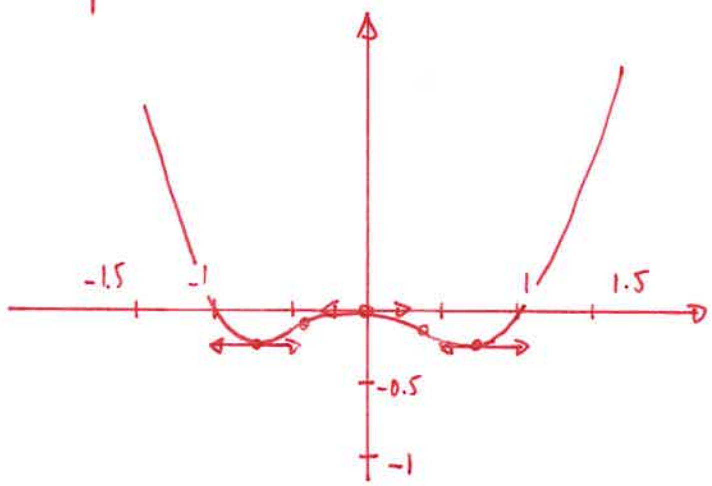
So $f''(x) = 0$ for $x = \pm\frac{1}{\sqrt{6}} \sim \pm 0.4$

At $x = \frac{1}{\sqrt{6}}$ $f\left(\frac{1}{\sqrt{6}}\right) = \frac{1}{36} - \frac{1}{6} = \frac{-5}{36}$ At $x = \frac{-1}{\sqrt{6}}$ $f\left(\frac{-1}{\sqrt{6}}\right) = \frac{-5}{36} \sim -0.14$

d)

x	-1.5	-1	-0.7	-0.4	0	0.4	0.7	1	+1.5
$f'(x)$		$-$	0	+0.5	0	-0.5	0	+	
$f''(x)$				0		0			
Shape	↘			↗		↘		↗	

$f'\left(\frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}}\left(2\times\frac{1}{6}-1\right) = \frac{-4}{3\sqrt{6}}$
~ -0.5



e) f is concave down for $\frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

THE SECOND DERIVATIVE AND TURNING POINTS

13 The revenue function for a magazine is given by $R = 4500x - 500x^2$, where x is the cost per issue of the magazine. What will be the cost per issue of the magazine to achieve maximum revenue?

$R'(x) = 4500 - 1000x$

So the maximum revenue will be achieved for $4500 - 1000x = 0$

$x = \frac{45}{10} = 4.5$

For the cost of $x = 4.5$, the revenue will be

$R(4.5) = 4500 \times 4.5 - 500 \times 4.5^2$

$R(4.5) = 10,125$

14 The revenue equation for a manufacturer is $R = \frac{80x - x^2}{4}$, where x is the number of units sold. How many units must be sold to achieve maximum revenue?

$R(x) = 20x - \frac{x^2}{4}$ $\qquad R'(x) = 20 - \frac{x}{2}$

So $R'(x) = 0$ for $x = 40$

For $x = 40$ $\qquad R(40) = \frac{80 \times 40 - 40^2}{4} = 400$

The maximum revenue is achieved for 40 units sold.

THE SECOND DERIVATIVE AND TURNING POINTS

15 A supplier has a monopoly on sales of books. The supplier's profit function is given by $P = 396x - 2.2x^2 - 400$, where x is the number of books sold.

(a) How many books must the supplier sell to maximise the profit?

(b) What is the maximum profit?

(c) If the government imposes a new 'monopoly tax' of $22 per book on the supplier, what is the new profit equation?

(d) Under the monopoly tax, how many books must the supplier now sell to maximise the profit? What is the new maximum profit?

a) $P(x) = 396x - 2.2x^2 - 400$

$P'(x) = 396 - 4.4x$

So $P'(x) = 0$ for $x = 396/4.4 = 90$ books.

b) for $x = 90$ $R(x) = 396 \times 90 - 2.2 \times 90^2 - 400 = 17{,}420$

c) We need to subtract $22x$ from the original profit:

$NP(x) = 396x - 2.2x^2 - 400 - 22x$

$NP(x) = 374x - 2.2x^2 - 400$

d) $NP'(x) = 374 - 2.2 \times 2 \times x = 374 - 4.4x$

so $NP'(x) = 0$ for $x = 374/4.4 = 85$ books

The new maximum profit is

$NP(85) = 374 \times 85 - 2.2 \times 85^2 - 400$

$NP(85) = 15{,}495$