SCALAR PRODUCT OF VECTORS IN THREE DIMENSIONS

The scalar product (dot product) has been defined and used for vectors in two dimensions. (Refer to *New Senior Mathematics Extension 1*, Section 10.4.)

In component form, it is written $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right) = x_1x_2 + y_1y_2$.

In three dimensions, if $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, then

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}\right) = x_1x_2 + y_1y_2 + z_1z_2.$$

The proof of this result is similar to the proof in two dimensions.

$$\begin{aligned}
\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}\right) \\
&= (x_1x_2)(\underset{\sim}{i} \bullet \underset{\sim}{i}) + (x_1y_2)(\underset{\sim}{i} \bullet \underset{\sim}{j}) + (x_1z_2(\underset{\sim}{i} \bullet \underset{\sim}{k})) + (y_1x_2)(\underset{\sim}{j} \bullet \underset{\sim}{i}) + (y_1y_2)(\underset{\sim}{j} \bullet \underset{\sim}{j}) + (y_1z_2)(\underset{\sim}{j} \bullet \underset{\sim}{k}) \\
&\quad + (z_1x_2)(\underset{\sim}{k} \bullet \underset{\sim}{i}) + (z_1y_2)(\underset{\sim}{k} \bullet \underset{\sim}{j}) + (z_1z_2)(\underset{\sim}{k} \bullet \underset{\sim}{k}) \\
&= x_1x_2 + y_1y_2 + z_1z_2 \text{ as } \underset{\sim}{i} \bullet \underset{\sim}{i} = \underset{\sim}{j} \bullet \underset{\sim}{j} = \underset{\sim}{k} \bullet \underset{\sim}{k} = 1 \text{ and } \underset{\sim}{i} \bullet \underset{\sim}{j} = \underset{\sim}{i} \bullet \underset{\sim}{k} = \underset{\sim}{j} \bullet \underset{\sim}{k} = \underset{\sim}{j} \bullet \underset{\sim}{i} = \underset{\sim}{k} \bullet \underset{\sim}{i} = \underset{\sim}{k} \bullet \underset{\sim}{j} = 0.
\end{aligned}$$

It is sometimes useful to be able to use sigma notation for the scalar product.

If $\underset{\sim}{u} = x_1\underset{\sim}{i} + x_2\underset{\sim}{j} + x_3\underset{\sim}{k}$ and $\underset{\sim}{v} = y_1\underset{\sim}{i} + y_2\underset{\sim}{j} + y_3\underset{\sim}{k}$, then $\underset{\sim}{u} \bullet \underset{\sim}{v} = x_1y_1 + x_2y_2 + x_3y_3 = \sum_{i=1}^{3} x_iy_i$.

Example 6

Given $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, find:

(a) $\underset{\sim}{a} \infty \underset{\sim}{b}$ **(b)** $\underset{\sim}{b} \infty \underset{\sim}{a}$ **(c)** $|\underset{\sim}{a}|$ **(d)** $\hat{\underset{\sim}{a}}$

Solution

(a)
$$\begin{aligned}
\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \\
&= 2 \times 1 + (-1) \times 2 + 3 \times (-1) \\
&= 2 - 2 - 3 \\
&= -3
\end{aligned}$$

(b)
$$\begin{aligned}
\underset{\sim}{b} \bullet \underset{\sim}{a} &= \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \bullet \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \\
&= 1 \times 2 + 2 \times (-1) + (-1) \times 3 \\
&= 2 - 2 - 3 \\
&= -3
\end{aligned}$$

(c) $|\underset{\sim}{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$

(d) $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \dfrac{1}{\sqrt{14}}\left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$

SCALAR PRODUCT OF VECTORS IN THREE DIMENSIONS

Angle between two vectors

Just as the scalar product can be used to find the angle between two-dimensional vectors, it can also be used to find the angle between two vectors in three dimensions.

The scalar product says $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, where θ is the angle between the two vectors.

Hence

$$\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \times \sqrt{x_2^2 + y_2^2 + z_2^2}}$$

Example 7

Find the angle, in degrees, between the vectors $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

Solution

$\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$: $|\underset{\sim}{a}| = \sqrt{4+1+1} = \sqrt{6}$

$\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$: $|\underset{\sim}{b}| = \sqrt{1+4+1} = \sqrt{6}$

$$\begin{aligned}\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(-2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \\ &= (-2)\times 1 + (-1)\times 2 + (-1)\times(-1) \\ &= -3\end{aligned}$$

$$\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{-3}{6} = -\frac{1}{2}$$

As $\cos\theta$ is negative, the angle must be obtuse, and hence in the second quadrant so $\theta = 120°$.

Example 8

Show that the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{v} = 5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ are perpendicular to each other.

Solution

Consider $u \bullet v$:

$$\begin{aligned}\underset{\sim}{u} \bullet \underset{\sim}{v} &= \left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right) \bullet \left(5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \\ &= 10 - 6 - 4 \\ &= 0\end{aligned}$$

Since the scalar product is zero, the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular. (Remember, $\cos 90° = 0$.)

Example 9

Find a vector perpendicular to the vector $3\underset{\sim}{i} - 4\underset{\sim}{j}$.

Solution

Let a unit vector perpendicular to $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ be $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

Since $\hat{\underset{\sim}{u}}$ is a unit vector, then $\quad x^2 + y^2 = 1 \quad [1]$

Since $\underset{\sim}{v} \bullet \hat{\underset{\sim}{u}} = 0$: $\quad 3x - 4y = 0 \quad [2]$

[2] becomes $\quad y = \frac{3x}{4}$

Substitute in [1]: $\quad x^2 + \frac{9x^2}{16} = 1$

$$25x^2 = 16$$

$$x = \pm\frac{4}{5}, \quad y = \pm\frac{3}{5}$$

Hence $\frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ and $-\frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ are unit vectors perpendicular to $3\underset{\sim}{i} - 4\underset{\sim}{j}$.

Since any scalar multiples of these unit vectors are also perpendicular to $\underset{\sim}{v}$, two possible answers are $4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $-\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$.

Algebraic properties of the dot product

These results have been used before in the Mathematics Extension 1 course.

- $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$
- $\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}|^2$
- $\underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$
- $(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{c} + \underset{\sim}{d}) = \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{d} + \underset{\sim}{b} \bullet \underset{\sim}{c} + \underset{\sim}{b} \bullet \underset{\sim}{d}$
- $(m\underset{\sim}{a}) \cdot \underset{\sim}{b} = m(\underset{\sim}{a} \cdot \underset{\sim}{b})$ where m is a real number.

Geometric properties of the dot product

- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, $\underset{\sim}{a} \perp \underset{\sim}{b}$
- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors
- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$
- The angle between two vectors, θ, is given by $\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$, $\underset{\sim}{a} \neq 0$, $\underset{\sim}{b} \neq 0$
- By convention, $0° \leq \theta \leq 180°$.

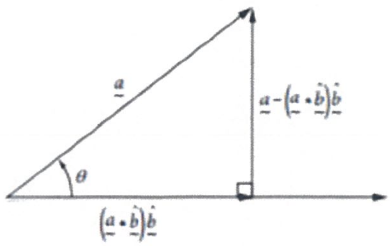
SCALAR PRODUCT OF VECTORS IN THREE DIMENSIONS

Scalar and vector projections of vectors

In *New Senior Mathematics Extension 1*, expressions were given for the scalar and vector projection of the two-dimensional vector, a, onto the two-dimensional vector, b.

- The scalar projection of a onto b is $a \cdot \hat{b}$, where $a \cdot \hat{b} = \dfrac{a \cdot b}{|b|}$.
- The vector projection of a onto b is $\left(a \cdot \hat{b}\right)\hat{b}$ or $\dfrac{a \cdot b}{b \cdot b}b$.
- The vector projection of a perpendicular to b is $a - \dfrac{a \cdot b}{b \cdot b}b$, or $a - \left(a \cdot \hat{b}\right)\hat{b}$.

These results also apply to three-dimensional vectors.



Example 10

Given the vectors $a = 4i + 5j - 3k$ and $b = 2i - 2j + k$, find:

(a) the scalar projection of a onto b

(b) the vector projection of a onto b

(c) the vector projection of a onto the x-axis. Hence write the vector projection on the y- and z-axes

(d) the vector projection of a perpendicular to b.

Solution

$b = 2i - 2j + k$: $|b| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$, $\hat{b} = \frac{1}{3}\left(2i - 2j + k\right)$

(a) Scalar projection

$$\text{onto } b = \frac{a \cdot b}{|b|} = \frac{\left(4i + 5j - 3k\right) \cdot \left(2i - 2j + k\right)}{3} = \frac{8 - 10 - 3}{3} = -\frac{5}{3}$$

(b) Vector projection

$$\text{onto } b = \left(a \cdot \hat{b}\right)\hat{b} = \frac{a \cdot b}{|b|}\hat{b} = -\frac{5}{3} \times \frac{1}{3}\left(2i - 2j + k\right) = \frac{5}{9}\left(-2i + 2j - k\right)$$

(c) i is the unit vector in the direction of the x-axis: vector projection $= \left(a \cdot i\right)i$

$$= \left(\left(4i + 5j - 3k\right) \cdot i\right)i = 4i$$

The vector projection of a on the y- and z-axes is $5j$ and $-3k$.

(d) Vector projection of a perpendicular to $b = a - \dfrac{a \cdot b}{b \cdot b}b$

$$= \left(4i + 5j - 3k\right) - \frac{\left(4i + 5j - 3k\right) \cdot \left(2i - 2j + k\right)}{9}\left(2i - 2j + k\right)$$

$$= \left(4i + 5j - 3k\right) - \frac{-5}{9}\left(2i - 2j + k\right)$$

$$= 4i + 5j - 3k + \frac{10}{9}i - \frac{10}{9}j + \frac{5}{9}k$$

$$= \frac{1}{9}\left(46i + 35j - 22k\right)$$

Example 11

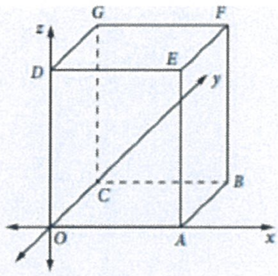
OABCDEFG is a cube of side length 1 unit. With reference to *O* as the origin:

(a) write the position vectors for $\overrightarrow{CB}$, $\overrightarrow{AB}$ and $\overrightarrow{CG}$

(b) find the position vectors of *F* and *G*

(c) find vectors $\overrightarrow{OF}$ and $\overrightarrow{AG}$

(d) calculate the acute angle at which the diagonals *OF* and *AG* intersect.



Solution

$\overrightarrow{OA} = \underset{\sim}{i}$, $\overrightarrow{OC} = \underset{\sim}{j}$ and $\overrightarrow{OD} = \underset{\sim}{k}$.

(a) $\overrightarrow{CB} = \overrightarrow{OA} = \underset{\sim}{i}$, $\overrightarrow{AB} = \overrightarrow{OC} = \underset{\sim}{j}$, $\overrightarrow{CG} = \overrightarrow{OD} = \underset{\sim}{k}$

(b) $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF}$

$= \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OG} = \overrightarrow{OC} + \overrightarrow{CG}$

$= \underset{\sim}{j} + \underset{\sim}{k}$

The position vector of *F* is $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$;

the position vector of *G* is $\underset{\sim}{j} + \underset{\sim}{k}$.

(c) $\overrightarrow{OF} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{AG} = \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CG}$

$= -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

(d) $\overrightarrow{OF} \bullet \overrightarrow{AG} = \left|\overrightarrow{OF}\right|\left|\overrightarrow{AG}\right|\cos\theta$

$\left|\overrightarrow{OF}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$,

$\left|\overrightarrow{AG}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) \bullet \left(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) = \sqrt{3}\sqrt{3}\cos\theta$

$-1 + 1 + 1 = 3\cos\theta$

$\cos\theta = \frac{1}{3}$

$\theta = 70°32$

The angle between the two vectors is 70°32′.

Example 12

Find unit vectors perpendicular to both $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{v} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

Solution

Let $\underset{\sim}{w} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ be a unit vector that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Use the fact that $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ when $\underset{\sim}{a} \perp \underset{\sim}{b}$.

$\underset{\sim}{u} \bullet \underset{\sim}{w} = 0$:	$2p - 3q + 6r = 0$	[1]
$\underset{\sim}{v} \bullet \underset{\sim}{w} = 0$:	$-6p + 2q + 3r = 0$	[2]
$\underset{\sim}{w}$ is a unit vector:	$p^2 + q^2 + r^2 = 1$	[3]
$[1] - 2 \times [2]$:	$14p - 7q = 0$	
	$q = 2p$	
$2 \times [1] + 3 \times [2]$:	$-14p + 21r = 0$	
	$r = \frac{2p}{3}$	
Substitute in [3]:	$p^2 + 4p^2 + \frac{4p^2}{9} = 1$	
	$49p^2 = 9$	
	$p = \pm\frac{3}{7}, \quad q = \pm\frac{6}{7}, \quad r = \pm\frac{2}{7}$	

Hence $\pm\frac{1}{7}\left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ are unit vectors perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Since any scalar multiple of these vectors is also perpendicular, then $\pm\left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ are vectors perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Since $\underset{\sim}{w}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$, then it is perpendicular to the plane containing $\underset{\sim}{u}$ and $\underset{\sim}{v}$.