CONTINUOUS PROBABILITY DISTRIBUTIONS

1 For the continuous random variable with probability density function:

$$f(x) = \begin{cases} \frac{3}{13}\left(x^2+4\right), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

find, correct to four decimal places:

(a) $P(X \le 0.75)$ **(b)** $P(X \le 0.9)$ **(c)** $P(X \ge 0.25)$ **(d)** $P(X \ge 0.65)$

a) $P(X \leqslant 0.75) = \int_0^{0.75} \frac{3}{13}(x^2+4)\,dx = \frac{3}{13}\int_0^{0.75}(x^2+4)\,dx$

$\_\_\_ = \frac{3}{13}\left[\frac{x^3}{3}+4x\right]_0^{0.75} = \frac{3}{13}\left[\frac{(0.75)^3}{3}+4\times 0.75\right] = \frac{603}{832} \simeq 0.7248$

b) $P(X \leqslant 0.9) = \int_0^{0.9} \frac{3}{13}(x^2+4)\,dx = \frac{3}{13}\left[\frac{0.9^3}{3}+4\times 0.9\right] \simeq 0.8868$

c) $P(X \geqslant 0.25) = 1 - P(X \leqslant 0.25) = 1 - \int_0^{0.25}\frac{3}{13}(x^2+4)\,dx$

$\_\_\_ = 1 - \frac{3}{13}\left[\frac{0.25^3}{3} + 4\times 0.25\right] \simeq 0.7680$

d) $P(X \geqslant 0.65) = 1 - P(X \leqslant 0.65)$

or $P(X \geqslant 0.65) = \int_{0.65}^{1}\frac{3}{13}(x^2+4)\,dx$ let's do it that way

$\_\_\_ = \frac{3}{13}\left[\left(\frac{1^3}{3}+4\times 1\right) - \left(\frac{0.65^3}{3}+4\times 0.65\right)\right]$

$\_\_\_ \simeq 0.3789$

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3 For the continuous random variable with probability density function:

$$f(x) = \begin{cases} \frac{3}{13}(x^2+4), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

find, correct to four decimal places, the following values:

(a) the mean

(b) the standard deviation.

a) $\mu = \int_0^1 x \times f(x)\,dx = \int_0^1 \frac{3x}{13}(x^2+4)\,dx = \frac{3}{13}\int_0^1 (x^3+4x)\,dx$

$\mu = \frac{3}{13}\left[\frac{x^4}{4} + 4\frac{x^2}{2}\right]_0^1 = \frac{3}{13}\left[\frac{1^4}{4} + 2\times 1^2\right] = \frac{27}{52} \simeq 0.5192$

b) $\sigma^2 = E(X^2) - [E(X)]^2$ is variance

$\sigma^2 = \int_0^1 x^2 f(x)\,dx - \left[\frac{27}{52}\right]^2$

We calculate $\int_0^1 x^2 f(x)\,dx = \frac{3}{13}\int_0^1 x^2(x^2+4)\,dx$

$\frac{3}{13}\int_0^1 (x^4+4x^2)\,dx = \frac{3}{13}\left[\frac{x^5}{5} + \frac{4x^3}{3}\right]_0^1 = \frac{3}{13}\left[\frac{1^5}{5} + \frac{4\times 1^3}{3}\right] = \frac{23}{65}$

So $\sigma^2 = \frac{23}{65} - \left[\frac{27}{52}\right]^2 = \frac{1139}{13520}$

So σ (standard deviation) is $\sqrt{\frac{1139}{13520}} \simeq 0.2903$

6 For the continuous random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{15000(x-50)}{x^4}, & x \geq 50 \\ 0, & \text{otherwise} \end{cases}$$

find the median, correct to the nearest whole number.

The median is the value m such that $P(X<m) = 0.5$

$$P(x<m) = \int_{50}^{m} \frac{15{,}000\,(x-50)}{x^4}\,dx = 15{,}000 \int_{50}^{m} \frac{x-50}{x^4}\,dx$$

$$— = 15{,}000 \int_{50}^{m} \left(\frac{1}{x^3} - \frac{50}{x^4}\right) dx = 15{,}000 \left[\frac{x^{-3+1}}{(-3+1)} - 50\,\frac{x^{-4+1}}{(-4+1)}\right]_{50}^{m}$$

$$— = 15{,}000 \left[-\frac{1}{2x^2} + \frac{50}{3x^3}\right]_{50}^{m}$$

$$— = 15{,}000 \left[\left(-\frac{1}{2m^2} + \frac{50}{3m^3}\right) - \left(-\frac{1}{2 \times 50^2} + \frac{50}{3 \times 50^3}\right)\right]$$

$$— = \frac{250{,}000}{m^3} - \frac{7{,}500}{m^2} + 15{,}000\left(\frac{1}{5000} - \frac{1}{7500}\right)$$

$$— = \frac{250{,}000}{m^3} - \frac{7{,}500}{m^2} + 1$$

We need to solve $\dfrac{250{,}000}{m^3} - \dfrac{7500}{m^2} + 1 = 0.5$

$$\Longleftrightarrow \frac{250{,}000}{m^3} - \frac{7{,}500}{m^2} + 0.5 = 0$$

By inspection, we find $m = 100$ which is the mean.

7 If $f(x) = \begin{cases} \dfrac{x}{45} + k, & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$

defines a probability density function, then the value of k is:

A 0 **B** $-\frac{1}{10}$ **C** $\frac{1}{10}$ **D** $\frac{1}{2}$

$$\int_0^6 \left(\frac{x}{45} + k\right) dx = \int_0^6 \frac{x}{45}\, dx + \int_0^6 k\, dx = \frac{1}{45}\int_0^6 x\, dx + 6k = \frac{1}{45}\left[\frac{x^2}{2}\right]_0^6 + 6k$$

$$\text{——} = \frac{1}{45} \times \frac{36}{2} + 6k = \frac{18}{45} + 6k$$

So for this to be equal to 1, we must have $\frac{18}{45} + 6k = 1$

$$\Leftrightarrow k = \frac{1}{6}\left[1 - \frac{18}{45}\right] = \frac{1}{10} \quad \text{Response } \boxed{C}$$

8 For the continuous random variable with probability density function:

$f(x) = \begin{cases} \dfrac{2x}{25}, & 0 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$

find, correct to four decimal places:

(a) $P(1.5 \le X \le 2.5)$

$$P(1.5 \leqslant X \leqslant 2.5) = \int_{1.5}^{2.5} \frac{2x}{25}\, dx = \frac{2}{25}\int_{1.5}^{2.5} x\, dx$$

$$\text{——} = \frac{2}{25}\left[\frac{x^2}{2}\right]_{1.5}^{2.5} = \frac{1}{25}\left[2.5^2 - 1.5^2\right]$$

$$\text{——} = 0.16$$

9 If $f(x) = \begin{cases} \frac{\pi}{12}\sin\left(\frac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$

then $P(X \le 3)$ is equal to:

A 0.45 $\boxed{\textbf{B}}$ 0.5 **C** 0.55 **D** 0.6

$$P(X \leqslant 3) = \int_0^3 \frac{\pi}{12} \sin\left(\frac{\pi x}{6}\right) dx = \frac{\pi}{12} \left[\left(\frac{6}{\pi}\right)\left(-\cos\left(\frac{\pi x}{6}\right)\right)\right]_0^3$$

$$= \frac{1}{2}\left[-\cos\left(\frac{\pi x}{6}\right)\right]_0^3$$

$$= \frac{1}{2}\left[\left[-\cos\left(\frac{\pi \times 3}{6}\right)\right] - \left[-\cos\left(\frac{\pi \times 0}{6}\right)\right]\right]$$

$$= \frac{1}{2}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{1}{2}(0+1) = \frac{1}{2}$$

Response $\boxed{B}$

10 For the uniform continuous random variable with probability density function:

$f(x) = \begin{cases} k, & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$

find the following values:

(a) k **(b)** $P(X \le 3)$ **(c)** $P(X \le 5)$

a) $k = 1/6$ as $\int_0^6 f(x)dx = 1$ so $\int_0^6 k\,dx = 1 \Leftrightarrow k \times 6 = 1$

b) $P(X \leqslant 3) = \int_0^3 \frac{1}{6}\,dx = \frac{1}{6} \times 3 = \frac{1}{2}$

c) $P(X \leqslant 5) = \int_0^5 \frac{1}{6}\,dx = \frac{1}{6} \times 5 = \frac{5}{6}$

12 In the javelin competition at a primary schools athletics carnival, it is found that the distance s metres that the javelin is thrown is a continuous random variable with probability density function:

$$f(s) = \begin{cases} \frac{1}{486}\left(81 - s^2\right), & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Find:

(a) the mean distance the javelin is thrown

(b) the standard deviation for the distance the javelin is thrown

(c) the median distance the javelin is thrown.

a) $\mu = \int_0^9 \frac{s}{486}(81 - s^2)\,ds = \frac{1}{486}\int_0^9 (81s - s^3)\,ds.$

$\mu = \frac{1}{486}\left[\frac{81s^2}{2} - \frac{s^4}{4}\right]_0^9 = \frac{1}{486}\left[\frac{81 \times 9^2}{2} - \frac{9^4}{4}\right] \approx 3.375\text{ m}$

b) $\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \int_0^9 \frac{s^2}{486}\left[81 - s^2\right] ds = \frac{1}{486}\left[\frac{81s^3}{3} - \frac{s^5}{5}\right]_0^9$

$E(X^2) = \frac{1}{486}\left[\frac{81 \times 9^3}{3} - \frac{9^5}{5}\right] = 16.2$

So $\sigma^2 = 16.2 - (3.375)^2 = 4.809375$ $\quad \sigma = 2.19$

c) $\int_0^m \frac{1}{486}(81 - s^2)\,ds = \frac{1}{486}\left[81s - \frac{s^3}{3}\right]_0^m = \left(81m - \frac{m^3}{3}\right) \times \frac{1}{486}$

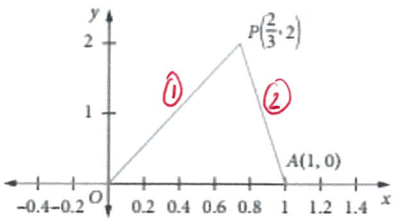
This is equal to 0.5 when $\frac{1}{486}\left(81m - \frac{m^3}{3}\right) = 0.5$

$\Leftrightarrow 81m - \frac{m^3}{3} = 243 \Leftrightarrow 243m - m^3 = 729$

Graphically, we find $m \approx 3.126\text{ m}$

CONTINUOUS PROBABILITY DISTRIBUTIONS

16 Consider the function f whose graph is shown.



(a) Show that the function could be used to represent a probability density function. Explain your answer using mathematical reasoning.

(b) Specify the rule of f.

(c) Consider a continuous random variable, X, with probability density function f. Calculate $P\left(X < \frac{2}{3}\right)$ and $P\left(X > \frac{2}{3}\right)$ using the areas of the triangles.

(d) Consider a continuous random variable X with probability density function f. Calculate $P\left(X < \frac{2}{3}\right)$ and $P\left(X > \frac{2}{3}\right)$ using integration. Comment on the results from parts **(c)** and **(d)**.

(e) Calculate $P\left(\frac{1}{2} < X < \frac{5}{6}\right)$, using integration.

a) line 1 has for equation $y = \frac{2}{2/3} x = 3x$

line 2 has for gradient $m = -\frac{2}{1 - 2/3} = -\frac{2}{1/3} = -6$

and for equation $y - 0 = -6(x - 1)$ so $y = -6x + 6$

Let's check that $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{2/3} 3x\,dx + \int_{2/3}^{1} (-6x+6)\,dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^{2/3} - 6\left[\frac{x^2}{2} - x\right]_{2/3}^{1}$$

$$= \frac{3}{2}\left(\frac{2}{3}\right)^2 - 6\left[\left(\left(\frac{1^2}{2}\right) - 1\right) - \left(\frac{(2/3)^2}{2} - \frac{2}{3}\right)\right]$$

$$= \frac{3}{2} \times \frac{4}{9} + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1 \text{ indeed.}$$

So it could be used to represent a probability density function.

b) $f(x) = \begin{cases} 3x & \text{if } 0 \leqslant x \leqslant 2/3 \\ -6(x-1) & \text{if } 2/3 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere.} \end{cases}$

c) $P(X<2/3) = \int_0^{2/3} 3x\,dx = 3\left[\frac{x^2}{2}\right]_0^{2/3} = \frac{3}{2}\times\left(\frac{4}{9}\right) = \frac{2}{3}$

$P(X>2/3) = \int_{2/3}^{1} -6(x-1)\,dx = -6\left[\frac{x^2}{2}-x\right]_{2/3}^{1} = 6\left[x-\frac{x^2}{2}\right]_{2/3}^{1}$

—— $= 6\left[\left(1-\frac{1^2}{2}\right)-\left(\frac{2}{3}-\frac{(2/3)^2}{2}\right)\right] = 6\left[\frac{1}{2}-\frac{2}{3}+\frac{2}{9}\right] = \frac{1}{3}$ indeed.

d) as above.

e) $P\left(\frac{1}{2}<x<\frac{5}{6}\right) = \int_{1/2}^{2/3} 3x\,dx + \int_{2/3}^{5/6} -6(x-1)\,dx$

—— $= \frac{3}{2}\left[\left(\frac{2}{3}\right)^2-\left(\frac{1}{2}\right)^2\right] - 6\left[\frac{x^2}{2}-x\right]_{2/3}^{5/6}$

—— $= \frac{3}{2}\left[\frac{7}{36}\right] - 6\left[\left(\frac{(5/6)^2}{2}-5/6\right)-\left(\frac{(2/3)^2}{2}-2/3\right)\right]$

—— $= \frac{7}{24} - 6\left[-\frac{35}{72}+\frac{4}{9}\right]$

—— $= \frac{13}{24}$