MULTIPLE ROOTS OF A POLYNOMIAL EQUATION

A polynomial of degree n has n zeros, but they are not necessarily all different. You say that c is a zero of multiplicity $r\,(r > 1)$ when the factor $(x - c)$ occurs r times.

For example, if $P(x) = (x-1)^3(x-5)^2(x-6)$, then 1 is a zero of multiplicity three, 5 is a zero of multiplicity two and 6 is a zero of multiplicity one.

Furthermore, if $x = c$ is a zero of multiplicity r of the real polynomial $P(x)$, then $x = c$ is also a zero of multiplicity $(r-1)$ of the derived polynomial $P'(x)$, a zero of multiplicity $(r-2)$ of the second derived polynomial $P''(x)$, and so on. The proof of this result follows.

If $P(x) = (x-c)^r S(x)$, where $r > 0$, $S(c) \neq 0$

then
$$\begin{aligned} P'(x) &= r(x-c)^{r-1} S(x) + (x-c)^r S'(x) \\ &= (x-c)^{r-1}[rS(x) + (x-c)S'(x)] \\ &= (x-c)^{r-1}Q(x) \quad [1] \end{aligned}$$

i.e. the polynomial $P'(x)$, has a zero $x = c$ of multiplicity $(r-1)$.

Applying the product rule to $P'(x)$ in [1] produces the polynomial $P''(x)$ with a zero $x = c$ of multiplicity $(r-2)$.

If $P(x)$ is a polynomial of degree n, then $P'(x)$ must be a polynomial of degree $(n-1)$, $P''(x)$ a polynomial of degree $(n-2)$, and so on. This property allows us to use calculus techniques to solve equations that are known to have multiple roots.

Example 15

Solve $x^3 - 4x^2 - 3x + 18 = 0$, given that it has a root of multiplicity 2.

Solution

Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$

Differentiate: $P'(x) = 3x^2 - 8x - 3$

Factorise: $P'(x) = (3x+1)(x-3)$

Solve: $P'(x) = 0$: $x = -\frac{1}{3}, 3$

As $P(x) = 0$ has a root of multiplicity 2 (i.e. a double root), the solution must be either $x = -\frac{1}{3}$ *or* $x = 3$, but not both.

Evaluate: $P\left(\frac{1}{3}\right) = -\frac{1}{27} - \frac{4}{9} + 1 + 18 = 17\frac{14}{27} \neq 0$

$P(3) = 27 - 36 - 9 + 18 = 0$. Thus $x = 3$ is a double zero.

$\therefore P(x) = (x-3)(x-3)(x-a)$

As $P(0) = 18$: $(-3) \times (-3) \times (-a) = 18$

$a = -2$

Thus $P(x) = (x-3)^2(x+2)$

The roots of the equation are $x = -2, 3, 3$.