2 Find the coordinates of the turning point of the curve $y = xe^{-0.5x}$ and state whether it is a maximum or minimum. Find the values of x for which:

(a) $y > 0$ **(b)** $\frac{dy}{dx} > 0$

$f(x) = x e^{-0.5x} = u(x)\, v(x)$ $\quad u(x) = x \quad u'(x) = 1$

$v(x) = e^{-0.5x} \quad v'(x) = -0.5 e^{-0.5x}$

So $f'(x) = e^{-0.5x} - x e^{-0.5x} \times 0.5 = e^{-0.5x}\left[1 - \frac{x}{2}\right]$

$f'(x) = 0$ when $x = 2$

At $x = 2$ $\quad f(2) = 2e^{-0.5 \times 2} = \frac{2}{e} \sim 0.74$

For $x = 0$ $\quad f(x) = 0$ $\quad$ So the point $\left(2, \frac{2}{e}\right)$ is a maximum.

a) $f(x) > 0$ for $x e^{-0.5x} > 0$ $\quad e^x > 0$ so

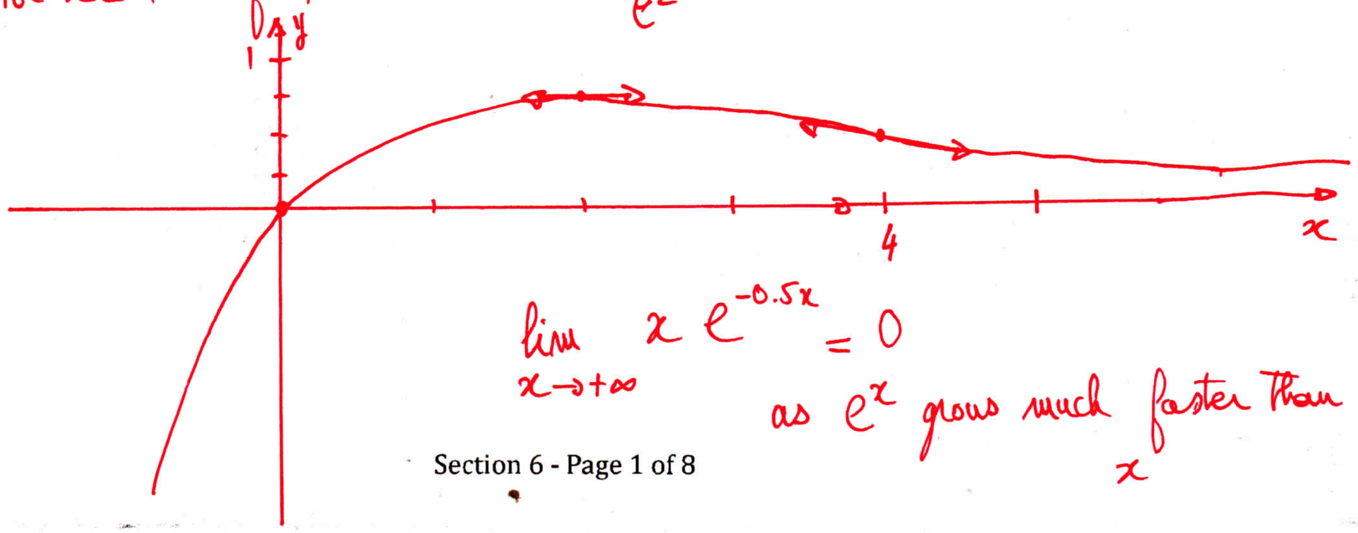
we must have $x > 0$ for $f(x)$ to be positive.

b) $f'(x) > 0 \Leftrightarrow e^{-0.5x}\left[1 - \frac{x}{2}\right] > 0 \Leftrightarrow 1 - \frac{x}{2} > 0 \Leftrightarrow x < 2$

$f''(x) = -0.5 e^{-0.5x}\left(1 - \frac{x}{2}\right) + \left(-\frac{1}{2}\right) e^{-0.5x} = e^{-0.5x}\left[-0.5 + \frac{x}{4} - 0.5\right]$

$f''(x) = e^{-0.5x}\left[\frac{x}{4} - 1\right]$ $\quad$ so $f''(x) = 0$ for $x = 4$ (inflection point)

For $x = 4$ $\quad f(4) = 4e^{-0.5 \times 4} = \frac{4}{e^2} \sim 0.54$



$\lim_{x \to +\infty} x e^{-0.5x} = 0$

as e^x grows much faster than x

3 Consider the function defined by the rule $f(x) = 3 - e^{-x}, x \geq 0$.

(a) Find the value of $f(0)$ and $f'(0)$.

(b) Show that $f'(x) > 0$ for all values of x in the domain.

(c) What is the value of $\lim_{x \to \infty} f(x)$?

(d) Sketch the graph of $f(x)$.

a) $f'(x) = e^{-x}$ $\quad f(0) = 3 - e^0 = 2$ $\quad f'(0) = e^0 = 1$

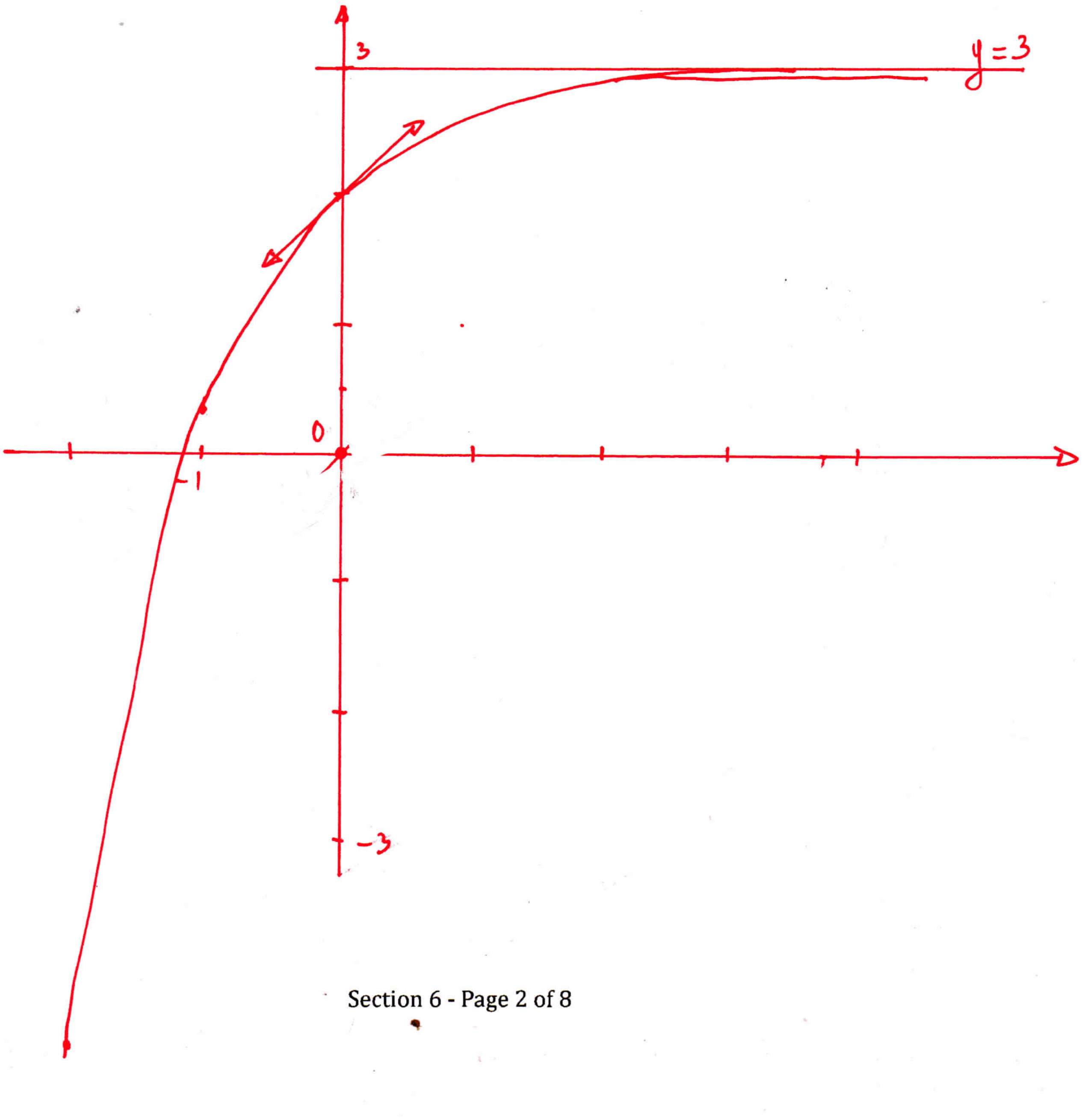
b) $e^{-x} > 0 \quad \therefore \quad f'(x) > 0$

$\forall x \in \mathbb{R}$

c) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (3 - e^{-x}) = 3$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (3 - e^{-x}) = -\infty$

d)



4 Consider the function defined by $f(x)=e^{-x^2}$ for all values of x.

(a) Find $f'(x)$.

(b) Find the values of x for which: (i) $f'(x)=0$ (ii) $f'(x)>0$ (iii) $f'(x)<0$.

(c) Sketch the graph of the function.

a) $f(x) = e^{-x^2}$ $\quad f'(x) = e^{-x^2} \times (-2x) = -2x\, e^{-x^2}$

b) i) $f'(x)=0 \Leftrightarrow -2x\, e^{-x^2} = 0 \Rightarrow x=0$ as $e^x \neq 0 \ \forall x \in \mathbb{R}$

ii) $f'(x)>0 \Leftrightarrow -2x\, e^{-x^2} > 0 \Rightarrow 2x\, e^{-x^2} < 0 \Rightarrow x<0$ as $e^x > 0 \ \forall x \in \mathbb{R}$

iii) $f'(x)<0 \Rightarrow x>0$

c)

x	$-\frac{1}{\sqrt2} = -0.7$	0	$\frac{1}{\sqrt2} = 0.7$
$f'(x)$	$+$	0	$-$
Shape	0 ↗ 0.61 inf ↗	1	↘ 0.61 inf ↘ 0

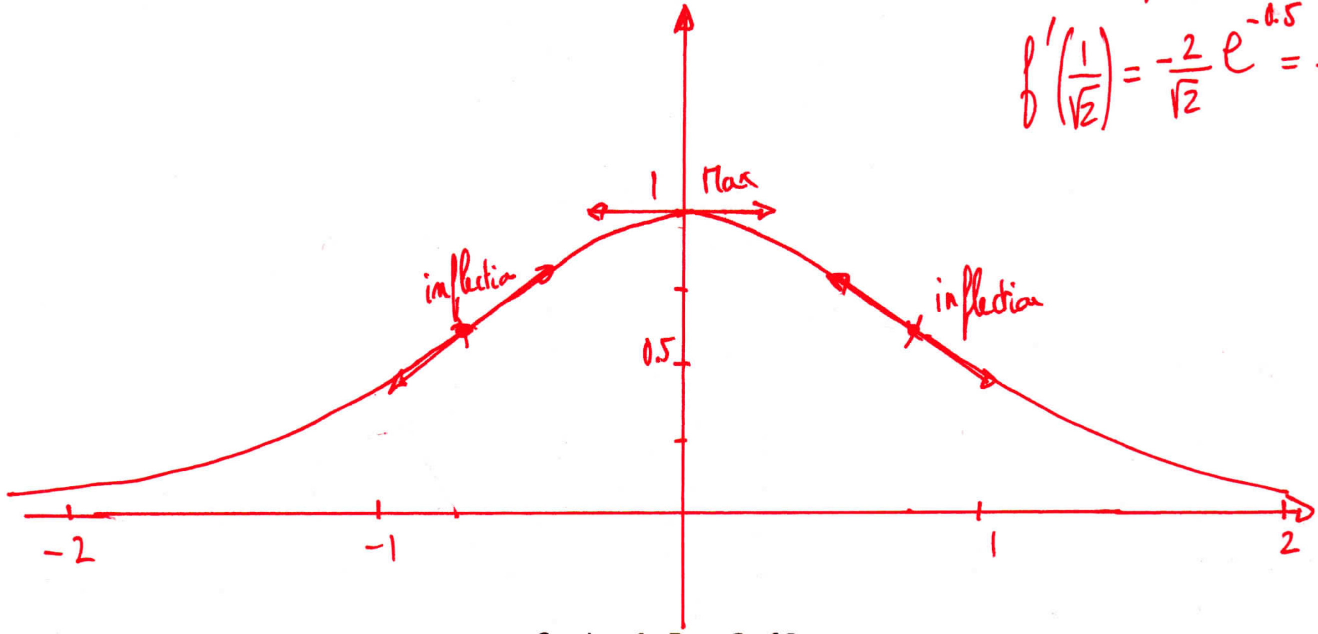
$\lim\limits_{x\to+\infty} e^{-x^2} = 0$

$\lim\limits_{x\to-\infty} e^{-x^2} = 0$

$f''(x) = -2e^{-x^2} - 2x\left(-2x e^{-x^2}\right) = e^{-x^2}\left[-2+4x^2\right] = 2e^{-x^2}\left[2x^2-1\right]$

So $f''(x)=0$ for $x = \pm\frac{1}{\sqrt2}$ $\quad f\left(\frac{\pm1}{\sqrt2}\right) = e^{-1/2} = \frac{1}{\sqrt e} \sim 0.61$

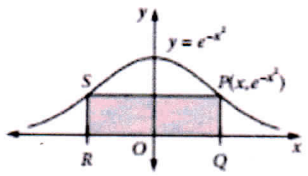
$f'\left(\frac{1}{\sqrt2}\right) = \frac{-2}{\sqrt2} e^{-0.5} = \frac{-2}{\sqrt{2e}} \sim -0.85$



8 The rectangle $PQRS$ has two vertices on the x-axis and two on the curve $y = e^{-x^2}$, as shown in the diagram. Find:

(a) the value of x for which the rectangle has a maximum area

(b) the maximum area of the rectangle.



a) Area $= 2x \times e^{-x^2} = f(x)$

$$f'(x) = 2e^{-x^2} + 2x\, e^{-x^2} \times (-2x) = 2e^{-x^2}\left[1-2x^2\right]$$

So PQRS is maximum(*) when $f'(x) = 0$, i.e. $x = \frac{1}{\sqrt{2}}$

b) for $x = \frac{1}{\sqrt{2}}$ Area $= 2 \times \frac{1}{\sqrt{2}} e^{-1/2} = \frac{2}{\sqrt{2e}}$ units^2

(or $\frac{\sqrt{2}}{\sqrt{e}}$ units^2)

(*) {As $\lim_{x \to \infty} 2xe^{-x^2} = 0$ we know it has to be a maximum (it cannot be a minimum)}

9 Find the coordinates of any maximum or minimum turning points on the curve $y = \frac{\ln x}{x}$.

x cannot be negative

$f(x) = \frac{\ln x}{x} = \frac{u(x)}{v(x)}$ $\quad u(x) = \ln x \quad u'(x) = 1/x$

$\quad v(x) = x \quad v'(x) = 1$

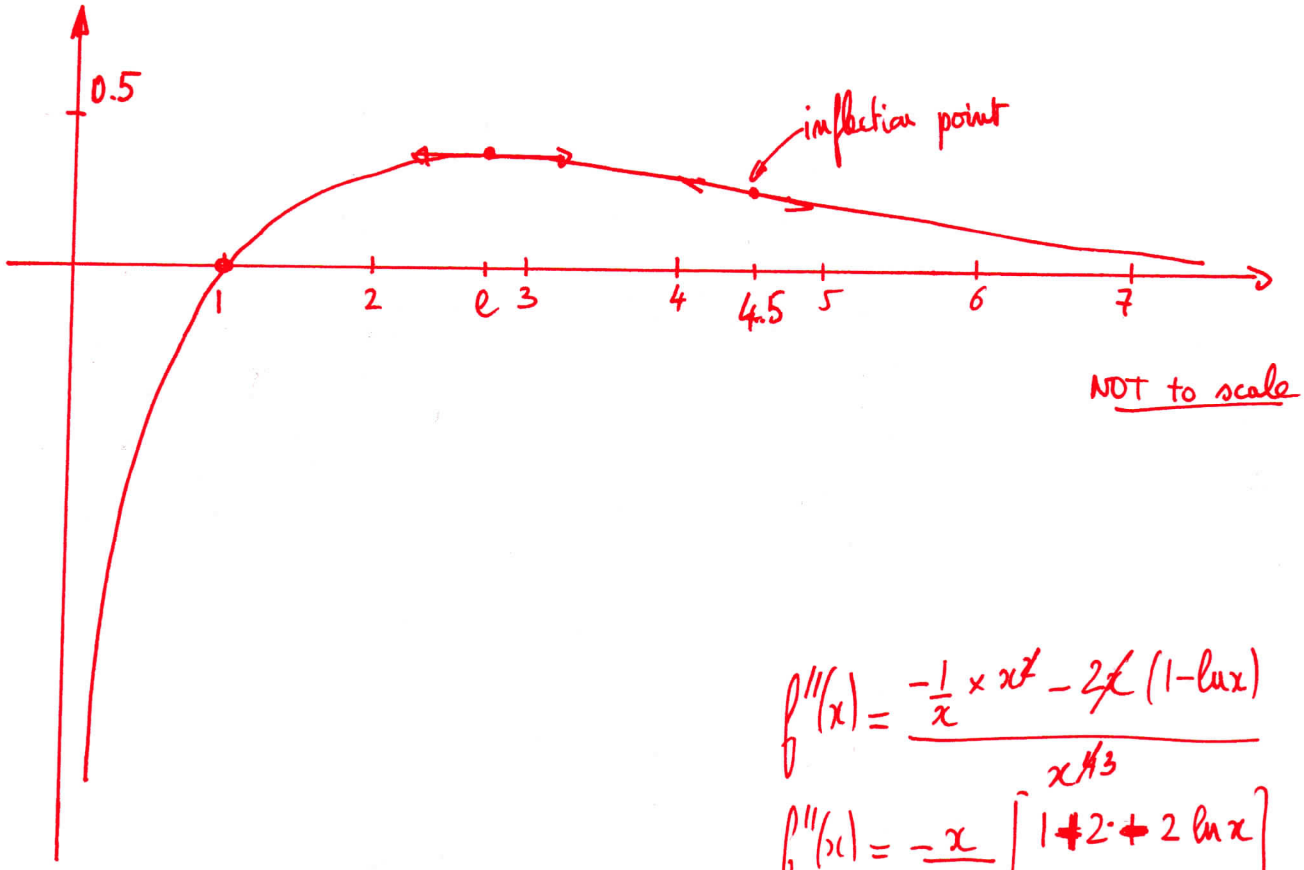
So $f'(x) = \frac{1 - \ln x}{x^2}$

$f'(x) = 0$ for $x = e$

When $x = e$ $\quad f(e) = \frac{1}{e} \sim 0.37$ so $\left(e, \frac{1}{e}\right)$ is the turning point

$f'(x) > 0$ for $x < e$ and $f'(x) < 0$ for $x > e$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\ln x}{x} = 0 \qquad \lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$



$f''(x) = \frac{-\frac{1}{x} \times \cancel{x^2} - 2\cancel{x}(1 - \ln x)}{x^{\cancel{4}3}}$

$f''(x) = -\frac{x}{x^4}\left[1 + 2 + 2\ln x\right]$

$f''(x) = -\frac{1}{x^3}\left[3 + 2\ln x\right]$

So $f''(x) = 0$ for $\ln x = +\frac{3}{2}$ $\quad x = e^{+3/2} \sim 4.5$

12 **(a)** Find all values of x between 0 and 2π for which $\log_e(\sin x)$ is defined.

(b) Find the maximum value of $\log_e(\sin x)$ and when it occurs.

a) $f(x) = \ln(\sin x)$ $\quad$ $\ln x$ can only exist for positive values of x

So $\sin x$ must be positive. so $0 < x < \pi$

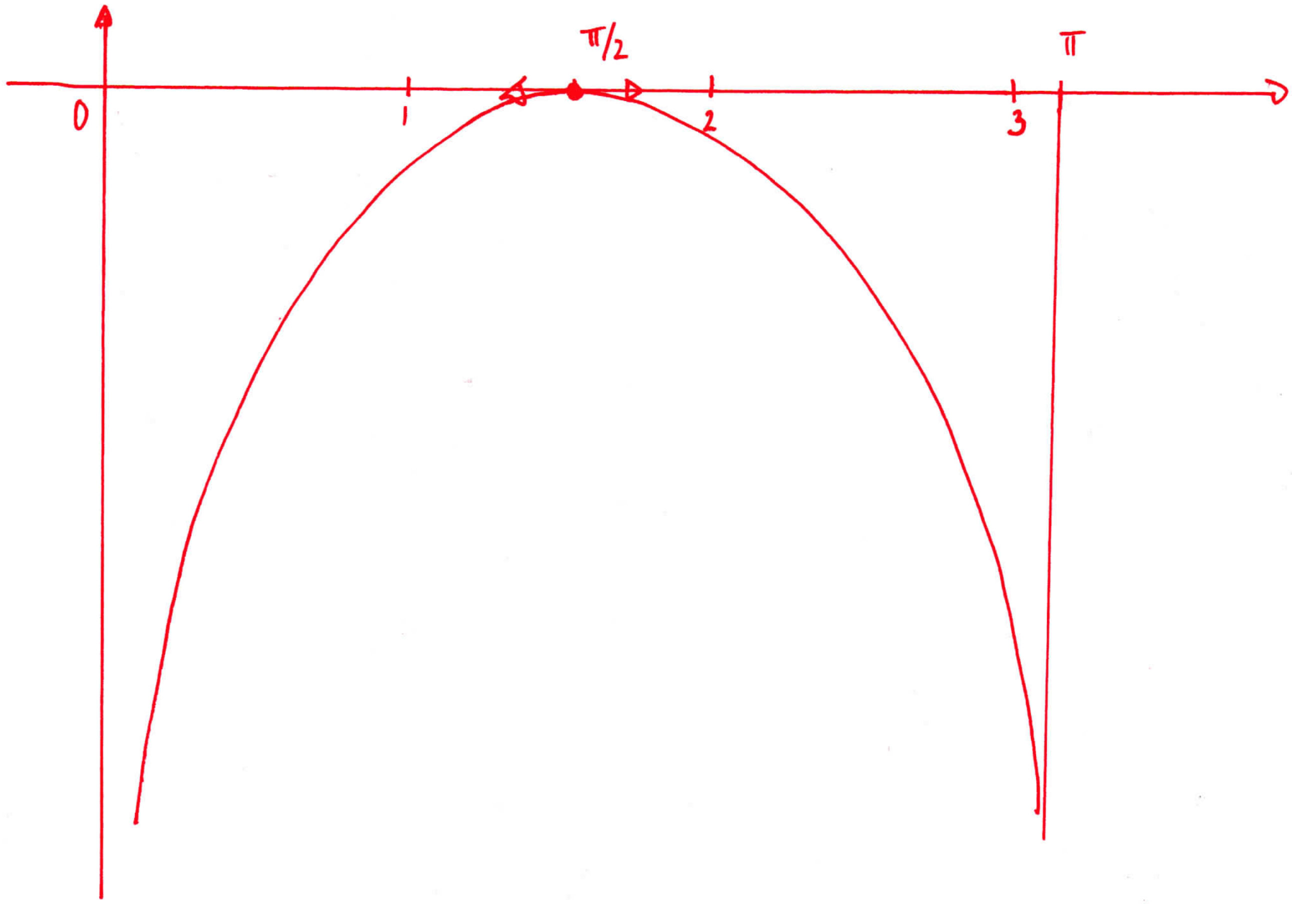
b) $f'(x) = \dfrac{1}{\sin x} \times \cos x = \cot x$

$\cot(x) = 0$ for $x = \frac{\pi}{2}$

At $x = \frac{\pi}{2}$ $\quad$ $f\left(\frac{\pi}{2}\right) = \ln\left(\sin\frac{\pi}{2}\right) = \ln 1 = 0$

$f'(x) > 0$ for $0 < x < \frac{\pi}{2}$ $\qquad$ $f'(x) < 0$ for $\frac{\pi}{2} < x < \pi$

$\lim\limits_{x \to 0^+} \ln(\sin x) = -\infty$ and $\lim\limits_{x \to \pi^-} \ln(\sin x) = -\infty$



14 When a uniform chain is suspended at two fixed points, it hangs in a catenary whose equation is $y = \frac{1}{2a}\left(e^{ax} + e^{-ax}\right)$

(a) Sketch the curve when $a = 0.5$ and the fixed points are at the same horizontal level and 8 units apart.

(b) Find the sag at the centre.

(c) Find the angle of inclination of the chain at the supports.

a) For $a = 0.5$ $\quad f(x) = e^{0.5x} + e^{-0.5x}$ which is an even function

At $x = 4$ $\quad f(4) = e^{2} + e^{-2} \sim 7.5$ and $f(-4) = e^{-2} + e^{2} = f(4)$

at $x = 0$ $\quad f(0) = 1 + 1 = 2$

b) So the sag in the centre is $(e^2 + e^{-2} - 2)$

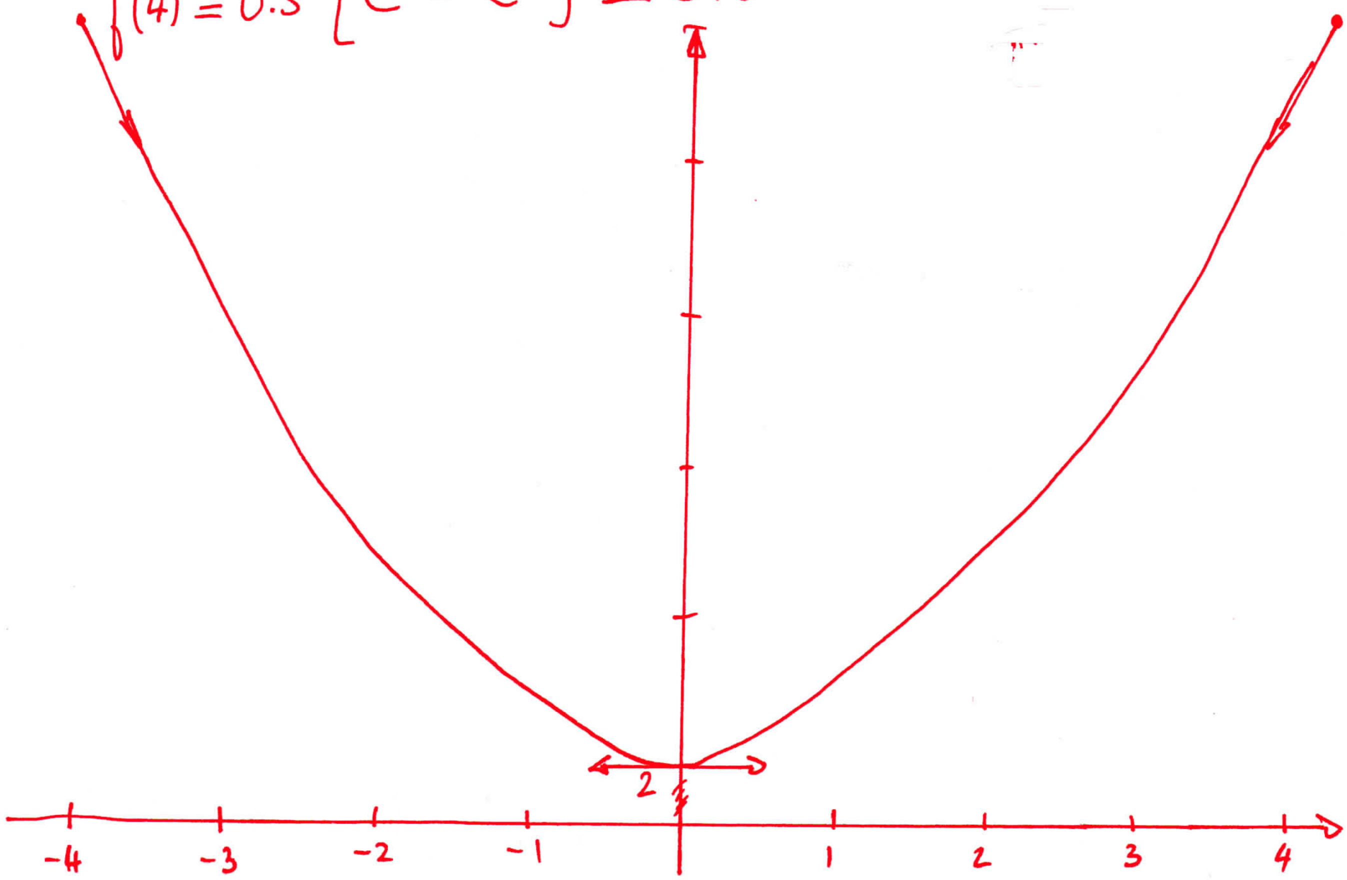
d) $f'(x) = 0.5\, e^{0.5x} - 0.5\, e^{-0.5x}$

$f'(x) = 0.5\left[e^{0.5x} - e^{-0.5x}\right]$

$f'(x) = 0$ for $x = 0$

At $x = \pm 4$ $\quad f'(-4) = 0.5\left(e^{-2} - e^{+2}\right) \sim -3.6$

$f'(4) = 0.5\left[e^{2} - e^{-2}\right] \simeq 3.6$



15 $f(x)$ is defined as $f(x) = e^{-x}\cos x$ in the domain $[0, \pi]$.

(a) Find $f(0)$, $f\left(\frac{\pi}{2}\right)$ and $f(\pi)$.

(b) Find $f'(x)$.

(c) Evaluate $f'(0)$ and $f'\left(\frac{3\pi}{4}\right)$.

(d) Sketch the graph of $y = f(x)$.

a) $f(0) = 1 \times 1 = 1$ $\quad f\left(\frac{\pi}{2}\right) = e^{-\pi/2} \times 0 = 0$ $\quad f(\pi) = e^{-\pi} \times (-1) = -e^{-\pi}$

b) $f'(x) = -e^{-x}\cos x - \sin x\, e^{-x} = -e^{-x}(\cos x + \sin x)$

c) $f'(0) = -1 \times (1+0) = -1$

$f'\left(\frac{3\pi}{4}\right) = -e^{-3\pi/4}\underbrace{\left(\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}\right)}_{=0} = 0$ $\quad f\left(\frac{3\pi}{4}\right) = e^{-3\pi/4} \times \left(-\frac{\sqrt{2}}{2}\right)$

d)

x	0		$\frac{\pi}{2}$		$\frac{3\pi}{4}$		π
$f'(x)$	-1	$-$			0	$+$	$e^{-\pi} \sim 0.04$
Shape	1		0		-0.07		$-e^{-\pi} \sim -0.04$