INFINITE GEOMETRIC SERIES

In Example 23 you observed that for $a = 4$, $r = \frac{1}{2}$:

$S_1 = 4, S_2 = 7, S_3 = 7, S_4 = 7\frac{1}{2}, S_5 = 7\frac{3}{4}, S_6 = 7\frac{7}{8}, S_7 = 7\frac{15}{16}, S_{10} = 7\frac{127}{128}$

It appears that as n increases, $S_n = 7$ + a fraction less than 1, so that as $n \to \infty$, $S_n \to 8$. Because the series approaches a limiting value, it is said that it **converges**.

You can define S_∞, the limiting sum of S_n, as $S_\infty = \lim_{n\to\infty} S_n$. Hence in this case $S_\infty = 8$.

Consider a piece of string 8 cm long. It is trimmed so that the first piece cut off is 4 cm long. The remainder is then cut in half so that the next piece is 2 cm long. The remainder is then cut in half so that the next piece is 1 cm long, and so on. You can imagine that there will always be a piece left over to be cut in half again, however small; but as the number of pieces cut off becomes increasingly large, their total length will get closer and closer to 8. That is:

$$S_\infty = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 8$$

You have seen that $S_n = \frac{a\left(1 - r^n\right)}{1 - r}$, which can be written as $S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$.

If $|r| < 1$, i.e. r is between -1 and 1, then $r^n \to 0$ as $n \to \infty$. In this example you are looking at $r = \frac{1}{2}$, so $r^n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$:

n	1	4	10	$\to \infty$
$\frac{1}{2^n}$	$\frac{1}{2}$	$\frac{1}{16}$	$\frac{1}{1024}$	$\to 0$

As $n \to \infty$, $\frac{1}{2^n} \to 0$. Thus $\lim_{n\to\infty} r^n = 0$ and so $\lim_{n\to\infty} \frac{ar^n}{1-r} = 0$, which means that the geometric series converges for $|r| < 1$:

$$S_\infty = \frac{a}{1-r} \qquad \text{for } |r| < 1$$

If $|r| > 1$ then $r^n \to \infty$ and there is no limiting sum, so the geometric series does **not** converge for $|r| > 1$.

Example 24

A rubber ball is dropped from a height of 20 m. Each time it strikes the ground it rebounds to $\frac{3}{4}$ of the height of the previous fall. Find the total number of metres it travels.

Solution

For the downward motion: $a = 20$, $r = \frac{3}{4}$, $n \to \infty$, $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{20}{1 - \frac{3}{4}} = 80$$

For the upward motion: $a = 15$, $r = \frac{3}{4}$, $n \to \infty$, $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{15}{1 - \frac{3}{4}} = 60$$

Total distance = 80 m + 60 m = 140 m

Example 25

Find the values of x for which the series $1 + (x-3)^2 + (x-3)^4 + \ldots$ converges.

Solution

For the series to be geometric, $a = 1$ and $r = (x-3)^2$.

Series converges for $|r| < 1$:

$$\begin{aligned} (x-3)^2 &< 1 \\ (x-3)^2 - 1 &< 0 \\ (x-3-1)(x-3+1) &< 0 \\ (x-4)(x-2) &< 0 \\ \therefore\ 2 < x &< 4 \end{aligned}$$

Example 26

Express as a rational number: **(a)** $0.\dot{2}\dot{3}$ **(b)** $0.5\dot{7}$

Solution

(a) $0.\dot{2}\dot{3} = 0.23 + 0.0023 + 0.000\,023 + \ldots$

Geometric series with $a = 0.23$, $r = \frac{1}{100}$, $S_\infty = \frac{a}{1-r}$

$$0.\dot{2}\dot{3} = \frac{0.23}{1-\frac{1}{100}}$$

$$= \frac{23}{99}$$

(b) $0.5\dot{7} = 0.5 + 0.07 + 0.007 + 0.0007 + \ldots$

Geometric series with $a = 0.07$, $r = 0.1$, $S_\infty = \frac{a}{1-r}$

$$0.5\dot{7} = 0.5 + \frac{0.07}{1-0.1}$$

$$= \frac{1}{2} + \frac{7}{90} = \frac{52}{90} = \frac{26}{45}$$