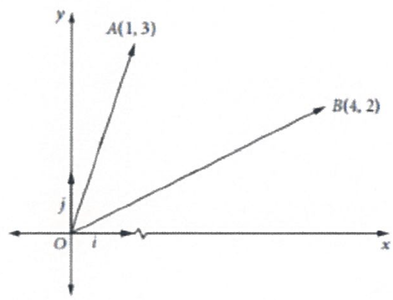
CARTESIAN COORDINATES IN THREE-DIMENSIONAL SPACE

You have used the Cartesian system for vectors in two dimensions.

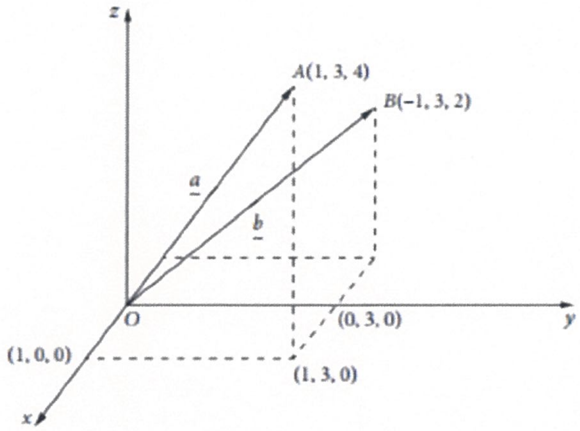
In the diagram, the vectors can be written in several forms,

such as $\overrightarrow{OA} = (1,3) = \underset{\sim}{i} + 3\underset{\sim}{j} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = (4,2) = 4\underset{\sim}{i} + 2\underset{\sim}{j} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.



A similar approach has been used in three-dimensional space. In this diagram, $\overrightarrow{OA} = \underset{\sim}{a} = (1, 3, 4) = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{j} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

and $\overrightarrow{OB} = \underset{\sim}{b} = (-1, 3, 2) = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.



Example 17

(a) Show by calculation that the points $A(1, -1, 3)$, $B(2, -4, 5)$ and $C(5, -13, 11)$ are collinear.

(b) Using vectors, show that ABC is a straight line.

Solution

(a) $AB = \sqrt{(2-1)^2 + (-4+1)^2 + (5-3)^2} = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$

$BC = \sqrt{3^2 + 9^2 + 6^2} = \sqrt{126} = 3\sqrt{14}$

$AC = \sqrt{4^2 + 12^2 + 8^2} = \sqrt{224} = 4\sqrt{14}$

$AB + BC = \sqrt{14} + 3\sqrt{14} = 4\sqrt{14} = AC$

Since the length of AC is the sum of the lengths on AB and BC, and the two intervals have the point B in common, then A, B and C are collinear.

(b) $\overrightarrow{AB} = (2-1, -4+1, 5-3) = (1, -3, 2)$

$\overrightarrow{BC} = (5-2, -13+4, 11-5) = (3, -9, 6) = 3(1, -3, 2)$

Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and they have a point, B, in common therefore ABC is a straight line.

Equation of a sphere

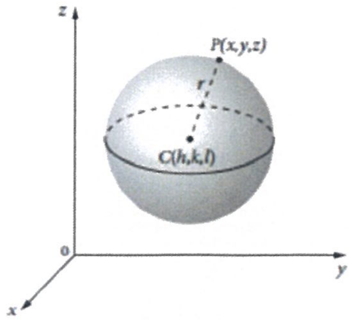
A circle is defined as the set of points in a plane equidistant from a fixed point in the plane. The equation of a circle centre (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$.

A sphere is defined as the set of points in three-dimensional space equidistant from a fixed point in space. If the point $P(x, y, z)$ is a point in space and $C(h, k, l)$ is the fixed point in space, then $PC = \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2}$ using the distance formula.

Let $PC = r$ so that $\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

which is the equation of the sphere centre (h, k, l) with radius, r.



Any plane that intersects the sphere will do so in a circle. In particular, the plane $z = l$ intersects the sphere in the circle $(x-h)^2 + (y-k)^2 = r^2$.

Example 18

Show that $x^2 + y^2 + z^2 + 6x - 4y + 2z + 6 = 0$ is the equation of a sphere and find the coordinates of its centre and its radius. Hence sketch this sphere.

Solution

$x^2 + y^2 + z^2 + 6x - 4y + 2z + 6 = 0$

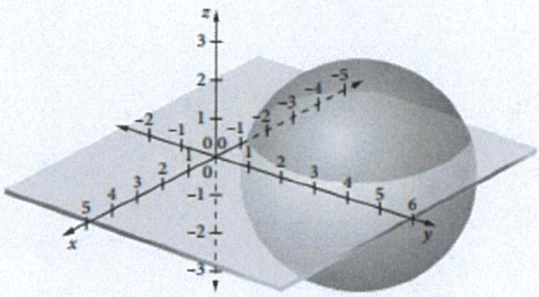
Rearrange the equation: $x^2 + 6x + y^2 - 4y + z^2 + 2z = -6$

Complete the square:

$x^2 + 6x + 9 + y^2 - 4y + 4 + z^2 + 2z + 1 = -6 + 9 + 4 + 1$

$(x+3)^2 + (y-2)^2 + (z+1)^2 = 8$

This is a sphere with centre $(-3, 2, -1)$ and radius $2\sqrt{2}$.



Example 19

The spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + (z-4)^2 = 16$ intersect. Find:

(a) the value of z when they intersect

(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

Solution

(a) Solve the equations simultaneously by subtracting the first equation from the second equation:

$x^2 + y^2 + (z-4)^2 - (x^2 + y^2 + z^2) = 16 - 9$

$z^2 - 8z + 16 - z^2 = 7$

$8z = 9$

$z = \frac{9}{8}$, so they intersect on the horizontal plane $z = \frac{9}{8}$.

(b) Substitute into the first equation: $x^2 + y^2 + \frac{81}{64} = 9$

$$x^2 + y^2 = \frac{495}{64}$$

$$x^2 + y^2 = \frac{3\sqrt{55}}{8}^{\,2}$$

The circle has centre $\left(0, 0, \frac{9}{8}\right)$, and radius $\frac{3\sqrt{55}}{8}$.