THE LANGUAGE AND LOGIC OF PROOF

1 Determine the negation of each of the following statements.

(a) p and q are both even.

(b) $x > 5$ or $x < -5$.

(c) x is divisible by either 7 or 8.

(d) $x = 0$ or $y = 0$.

2 Translate the following statements into everyday language. Also, determine whether the statement is true or false, justifying your answer where appropriate.

(a) $\forall$ integers n, the number $2n + 3$ is odd.

(b) $\exists$ a real number x such that $\frac{1}{x} = x$.

(c) $\forall$ real numbers x, $x^2 > 0$.

(d) $\exists\, x \in \mathrm{R}$ such that $x^2 = -1$.

(e) $\forall\, n \in$ integers, the number $n(n + 1)$ is divisible by 3.

(f) $\forall$ real numbers x and y, $x - y > 0$.

(g) $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.

(h) $\exists$ a real number x such that $\forall$ real numbers y, $xy = y$.

3 Rewrite the following statements using the symbols ∀ and ∃. Also, determine whether the statement is true or false, justifying your answer where appropriate.

(a) The square of any integer is greater than the integer.
(b) There is a real number which, when multiplied by 5 gives an answer of 0.
(c) The sum of any two consecutive integers is odd.
(d) There is a real number equal to its square.
(e) The sum of the squares of any two real numbers is less than the product of the numbers.
(f) There is a special real number with the property that whenever another real number is divided by this special number, this other real number is obtained as a result.
(g) Every integer is divisible by at least one integer.

4 Determine the negation of each of the following statements. Also state whether the original statement or the negation is true, justifying your answer where appropriate.

(a) $\forall$ real numbers x, $x^2 > 0$.

(b) $\exists$ a real number x such that $x^2 = x$.

(c) $\forall$ positive integers n, $10n > n$.

(d) $\forall$ real numbers x, x is either positive or negative.

(e) $\exists$ an integer n such that $n \neq 0$ and $n^2 < 1$.

(f) $\forall$ integers n, either $(-1)^n = 1$ or $(-1)^n = -1$.

5 Rewrite the following statements using the implication symbol $\Rightarrow$.

(a) If $x > 3$, then $x^2 > 9$.

(b) If n is divisible by 9, then n is divisible by 3.

(c) $n > 5$ implies that $n > 4$.

(d) $7p$ is positive if $p > 3$.

(e) q is even if $2q$ is a perfect square.

(f) m is a multiple of 6 is a sufficient condition to conclude that m is divisible by 3.

(g) It is necessary that $x^2 > 2$ if $x < -2$.

(h) n is even and greater than 2 is a sufficient condition to conclude that n is not prime.

6 Write the converse, the contrapositive, and the negation of each of the following conditional statements. Determine whether each of the original, converse, contrapositive, and negation are true or false, justifying your answer where appropriate.

(a) If n is divisible by 20, then n is divisible by 5.
(b) If n is divisible by 3, then n^2 is divisible by 3.
(c) If $x > 7$, then $10x > 70$.
(d) If $xy = 0$, then either $x = 0$ or $y = 0$.

7 Rewrite the following statements using the logical equivalence symbol, $\Leftrightarrow$.

(a) n is even if and only if n^2 is even.

(b) $x + y = 0$ if and only if $x = -y$.

(c) n being even and divisible by 3 is necessary and sufficient for n to be divisible by 6.

9 Which statement is true?

A $\forall x \in R, \exists y \in R$ such that $xy = 6$.

B $\exists x \in R$ such that $\forall y \in R, xy = 6$.

C $\exists x \in R$ such that $\forall y \in R, x + y = 6$.

D $\forall x \in R, \exists y \in R$ such that $x + y = 6$.

10 The negation of the statement $\forall x \in R, \exists y \in R$ such that $x + y = 6$, is:

A $\forall x \in R, \exists y \in R$ such that $x + y \neq 6$.

B $\exists x \in R$ such that $\forall y \in R, x + y \neq 6$.

C $\forall x \in R, \forall y \in R, x + y \neq 6$.

D $\exists x \in R, \exists y \in R$ such that $x + y \neq 6$.

11 Write the negation of the following statement, where x represents a real number: $x > 0$ and $x < 10 \Rightarrow x \geq 0$ and $x \leq 10$. Also, determine whether the original or the negation is true.

12 Consider the following conjecture:
Start with any positive integer. If the integer is even, halve it. If the integer is odd, triple it and add one. Repeat this process. Eventually, the integer 1 will be obtained.
This is known as the '$3x + 1$' conjecture. It is yet to be proved but has been shown to be true for all integers up to roughly 10^{14}. Verify this conjecture for the following positive integers:

(a) 6 **(b)** 13 **(c)** 7