GEOMETRIC SEQUENCES

1 Which of the following are geometric sequences?

(a) 3, 6, 12, 24, … (b) $8, -2, \frac{1}{2}, -\frac{1}{8}, \ldots$ (c) 2, 5, 11, 23, … (d) $\frac{1}{9}, \frac{1}{3}, 1, 3, \ldots$ (e) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

a) $\frac{6}{3} = 2$ whereas $\frac{12}{6} = 2$ and $\frac{24}{12} = 2$ so geometric sequence

b) $\frac{-2}{8} = -\frac{1}{4}$; $\frac{1/2}{-2} = -\frac{1}{4}$; $\frac{-1/8}{1/2} = -\frac{1}{4}$ so geometric sequence

c) $\frac{5}{2} \neq \frac{11}{5}$ so not geometric sequence

d) $\frac{1/3}{1/9} = \frac{1}{3} \times 9 = 3$; $\frac{1}{1/3} = 3$; $\frac{3}{1} = 3$ so YES

e) $\frac{1/4}{1/2} = \frac{1}{2}$; $\frac{1/6}{1/4} = \frac{1}{6} \times 4 = \frac{2}{3}$ so NOT.

2 For the geometric sequence 1, 3, 9, 27, … find:

(a) the value of a (b) the value of r (c) the expression for T_n

(d) the 10th term (e) the value of k if $T_k = 6561$.

a) $a = 1$ b) $r = 3$ c) $T_n = 3T_{n-1}$

$T_n = 1 \times 3^{n-1} = 3^{n-1}$

d) $T_{10} = 3^{10-1} = 3^9 = 19683$

e) if $T_k = 6561 = 3^{k-1}$

so $\ln 6561 = (k-1)\ln 3$

$\therefore \quad k = 1 + \frac{\ln 6561}{\ln 3} = 9$

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4 Find the sixth term of the sequence 4, 6, 9, 13.5, … .

$\frac{6}{4} = \frac{3}{2} = \frac{9}{6} = \frac{13.5}{9}$ so $r = \frac{3}{2}$

$T_4 = 13.5$ so $T_6 = \frac{3}{2} \times \frac{3}{2} \times 13.5 = 30.375 = \frac{243}{8}$

5 Find the first term and the common ratio of the geometric sequence in which $T_3 = 25$ and $T_5 = 156.25$.

$T_5 = r^2 \times T_3$ so $r^2 = \frac{156.25}{25}$ $r = \pm\sqrt{\frac{156.25}{25}} = \pm 2.5$

$\therefore T_2 = \frac{T_3}{2.5} = \frac{25}{2.5} = 10$ and $T_1 = \frac{10}{2.5} = 4$

OR $T_2 = \frac{T_3}{(-2.5)} = -10$ and $T_1 = \frac{-10}{-2.5} = 4$

7 Find the first three terms of a geometric sequence in which:

(a) the sixth term is 160 and the seventh term is 320

(b) the fifth term is 4 and the eighth term is $-\frac{1}{2}$.

a) $T_6 = 160$ $T_7 = 320$ so $r = \frac{320}{160} = 2$

$\therefore T_n = T_1 \times 2^{n-1}$ so $T_1 = \frac{T_6}{2^5} = \frac{160}{32} = 5$

so $T_2 = 10$ and $T_3 = 20$

b) $T_5 = 4$ and $T_8 = -\frac{1}{2}$ so $T_8 = T_5 \times r^3$

$\therefore r^3 = \frac{-1/2}{4} = -\frac{1}{8}$

so $r = -1/2$

$T_n = T_1 \times r^{n-1}$ so $T_1 = \frac{T_5}{(-1/2)^4} = 64$ $T_2 = -32$

$T_3 = 16$

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8 If $2p + 1$, $5p$ and $12p - 4$ are successive terms of a geometric sequence, find the value of p.

$\frac{5p}{2p+1} = \frac{12p-4}{5p}$ so $25p^2 = (2p+1)(12p-4)$

$\Leftrightarrow 25p^2 - 24p^2 = 4p - 4$

$\Leftrightarrow p^2 - 4p + 4 = 0$

$\Leftrightarrow (p-2)^2 = 0$

so $p = 2$

$\boxed{\Uparrow}$ 10 For x and y it is given that 1, x and y form an arithmetic sequence while 1, y and x form a geometric sequence. Find the value of x and y.

$x = 1 + k$ $\quad y = x + k$ $\quad \frac{y}{1} = \frac{x}{y}$

so $y^2 = x$

$\therefore y = y^2 + k$

$\Leftrightarrow y = y^2 + (x-1) = y^2 + (y^2 - 1)$

so $2y^2 - y - 1 = 0$ $\quad \Delta = 1 + 4 \times 2 = 9$

$y = \frac{1+3}{4} = 1$ or $y = \frac{1-3}{4} = -\frac{1}{2}$

if $y = 1$ then $x = 1$ Sequence $1, 1, 1$ and $1, 1, 1$

if $y = -\frac{1}{2}$ then $x = \frac{1}{4}$ Sequence $1, \frac{1}{4}, \frac{-1}{2}$ and $1, \frac{-1}{2}, \frac{1}{4}$

12 The population of a certain town is 10 000. If the population decreases each year by 10% of the population in the preceding year, find the population in 5 years' time.

$P_1 = 10{,}000$ $\quad$ $P_n = 0.9 \times P_{n-1}$

$P_n = 10{,}000 \times 0.9^{n-1}$

$\therefore \quad P_5 = 10{,}000 \times 0.9^5 \simeq 5{,}905$

14 Given that $a, ar, ar^2, ar^3, \ldots$ is a geometric sequence, show that the sequence $\log a, \log ar, \log ar^2, \log ar^3, \ldots$ is arithmetic.

$T_n = a\, T_{n-1}$

$\log ar - \log a = [\log a + \log r] - \log a = \log r$

$\log ar^2 - \log ar = [\log a + 2\log r] - [\log a + \log r] = \log r$

Generally

$\log ar^k - \log ar^{k-1} = \log\left(\frac{ar^k}{ar^{k-1}}\right) = \log r.$

Therefore $\log a$, $\log ar$, $\log ar^2, \ldots$ is an arithmetic sequence.
and its common difference is $\log r$

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15 Find a number which, when added to each of 2, 6, 13 gives three numbers in geometric sequence.

We must have : $\frac{6+k}{2+k} = \frac{13+k}{6+k} \iff k^2+12k+36 = k^2+15k+26$

$\iff 10 = 3k$

so $k = \frac{10}{3}$

4 **16** In a homogeneous nutrient medium the number of bacteria present doubles every hour. If there are initially 200 bacteria present, how many will be present after:

(a) 4 hours **(b)** 10 hours?

(c) If there are N bacteria present after t hours, write down a formula that gives the number of bacteria present after t hours.

$T_n = 200 \times 2^{n-1}$

a) $T_5 = 200 \times 2^{5-1} = 3200$ b) $T_{11} = 200 \times 2^{11-1} = 204{,}800$

c) $N = 200 \times 2^t$

18 The population of a colony of wading birds is decreasing by 15% each year. The initial population is 30000 birds.

(a) Find how many birds remain after 5 years (to the nearest thousand).

(b) Find when the population is equal to 9000.

(c) Sketch the graph of the population as a function of time.

a) $W_n = 30{,}000 \times 0.85^{n-1}$

So $W_6 = 30{,}000 \times 0.85^5 \simeq 13{,}000$

b) $9000 = 30{,}000 \times 0.85^{n-1}$

so $0.85^{n-1} = \frac{9}{30} = \frac{3}{10}$

$\therefore (n-1) \times \ln(0.85) = \ln(3/10)$

$\therefore n = 1 + \frac{\ln(3/10)}{\ln(0.85)} = 8.4$ so after 7.4 years.