REAL NUMBERS AND SURDS

Rational numbers may be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Integers are rational numbers with a denominator of 1, for example $5 = \frac{5}{1}$

Terminating decimals can be expressed as a fraction, therefore are rational numbers.

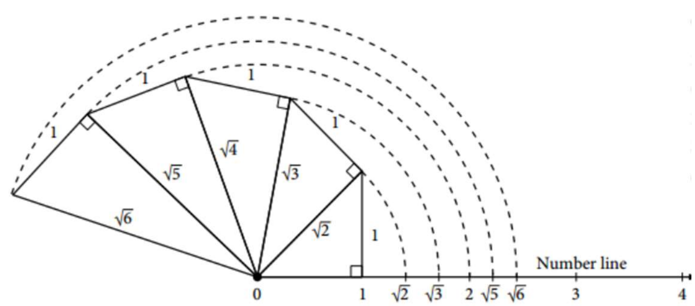
Example: $5.376 = \frac{5{,}376}{1{,}000}$

Recurring decimals can all be expressed as a fraction, therefore are rational numbers.

Example: let: $x = 5.33333\ldots = 5.\dot{3}$ therefore $10x = 53.33333\ldots$ and so:
$10x - x = 53.33333\ldots - 5.33333 = 48$ So $9x = 48$ and $x = \frac{48}{9} = \frac{16}{3}$
The same reasoning can be applied to any recurring decimal.

Irrational numbers are numbers that cannot be represented as a fraction such as $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}, 2\sqrt{7}$ or π. They are not rational. Irrational numbers that are roots, such as $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}$ or $2\sqrt{7}$ are called surds. Some other irrational numbers such as π are called transcendental numbers (meaning they cannot be the solution of a polynomial equation with rational coefficients).

The following diagram shows how to construct exact lengths for $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$ and $\sqrt{6}$, starting with a right-angle triangle with side lengths of one unit. Each triangle's hypotenuse is an exact value, written as a surd.



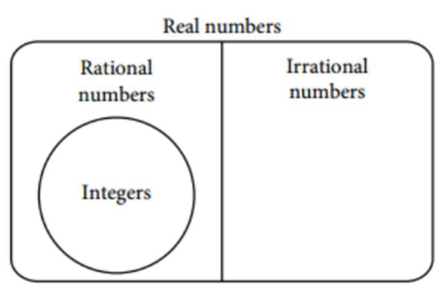
The diagram shows that $\sqrt{4} = 2$ which is a rational number.

The square root of a perfect square is always a rational number. Similarly, $\sqrt[3]{8} = 2$ is rational.

Surds that simplify to a rational number do not represent irrational numbers.

The sets of all rational and irrational numbers together form a larger set called the set of **real numbers.**

The real numbers can be represented by all the points on the real number line.



Operation with surds - Basic rules

Surds have their own set of simplifying rules for the four operations of arithmetic. These rules are used to simplify expressions involving surds.

1. $\sqrt{a} \times \sqrt{a} = a$ by definition of the square root of a number

2. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

   Proof: Squaring the LHS, we obtain: $\left(\sqrt{ab}\right)^2$ which is ab by definition of the square root of a number.
   Squaring the RHS, we obtain:
   $\left(\sqrt{a} \times \sqrt{b}\right)^2 = \left(\sqrt{a} \times \sqrt{b}\right) \times \left(\sqrt{a} \times \sqrt{b}\right) = \sqrt{a} \times \sqrt{b} \times \sqrt{a} \times \sqrt{b} = \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} = ab$
   Therefore, the squares of the LHS and the RHS are equal, and therefore the LHS and RHS must be equal, i.e. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

3. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

   Proof: Squaring the LHS, we obtain: $\left(\sqrt{\frac{a}{b}}\right)^2$ which is $\frac{a}{b}$ by definition of the square root of a number.
   Squaring the RHS, we obtain: $\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \left(\frac{\sqrt{a}}{\sqrt{b}}\right) \times \left(\frac{\sqrt{a}}{\sqrt{b}}\right) = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{a}}{\sqrt{b} \times \sqrt{b}} = \frac{a}{b}$
   Therefore, the squares of the LHS and the RHS are equal, and therefore the LHS and RHS must be equal, i.e. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Surds can also be simplified sometimes by identifying a factor in the surd that is a perfect square, and then taking that factor's square root. You might also multiply two surds and find that the answer has a perfect square factor.

You should know the perfect squares (i.e. 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, …) so that you can look for them as factors in surds. When a surd contains a perfect square factor, you can use the rules above to simplify the expression.

Example 15

Simplify each expression.

(a) $\sqrt{12}$ (b) $\sqrt{80}$ (c) $3\sqrt{28}$ (d) $\dfrac{\sqrt{175}}{5}$

(e) $\sqrt{6}\times\sqrt{10}$ (f) $\dfrac{\sqrt{80}}{\sqrt{5}}$ (g) $\sqrt{3}\times\sqrt{12}$ (h) $4\sqrt{2}\times\sqrt{24}$

Solution

(a) Look for a factor of 12 that is a perfect square.

$12 = 4\times 3$

$\therefore \sqrt{12} = \sqrt{4\times 3} = \sqrt{4}\times\sqrt{3} = 2\sqrt{3}$

(b) Look for a factor of 80 that is a perfect square.

$80 = 16\times 5$

$\therefore \sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5}$

(c) $3\sqrt{28} = 3\sqrt{4}\times\sqrt{7} = 6\sqrt{7}$

(d) $\dfrac{\sqrt{175}}{5} = \dfrac{\sqrt{25\times 7}}{5} = \dfrac{5\sqrt{7}}{5} = \sqrt{7}$

(e) $\sqrt{6}\times\sqrt{10} = \sqrt{60} = \sqrt{4\times 15} = 2\sqrt{15}$

(f) $\dfrac{\sqrt{80}}{\sqrt{5}} = \sqrt{\dfrac{80}{5}} = \sqrt{16} = 4$

(g) $\sqrt{3}\times\sqrt{12} = \sqrt{3}\times 2\sqrt{3} = 2\times 3 = 6$

(h) $4\sqrt{2}\times\sqrt{24} = 4\sqrt{2}\times 2\sqrt{6} = 8\sqrt{12} = 16\sqrt{3}$

Example 16

Simplify each expression, writing your answer with a rational denominator (or no denominator).

(a) $\dfrac{6}{\sqrt{3}}$ (b) $\dfrac{\sqrt{60}}{3\sqrt{6}}$ (c) $\dfrac{3\sqrt{5}}{\sqrt{15}}$

Solution

(a)
$$\begin{aligned}\frac{6}{\sqrt{3}} &= \frac{6}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{6\sqrt{3}}{3}\\ &= 2\sqrt{3}\end{aligned}$$

(b)
$$\begin{aligned}\frac{\sqrt{60}}{3\sqrt{6}} &= \frac{\sqrt{6}\times\sqrt{10}}{3\sqrt{6}}\\ &= \frac{\sqrt{10}}{3}\end{aligned}$$

(c)
$$\begin{aligned}\frac{3\sqrt{5}}{\sqrt{15}} &= \frac{3\sqrt{5}}{\sqrt{3}\times\sqrt{5}}\\ &= \frac{3}{\sqrt{3}}\\ &= \frac{\sqrt{3}\times\sqrt{3}}{\sqrt{3}}\\ &= \sqrt{3}\end{aligned}$$