APPROXIMATIONS OF TRIGONOMETRIC FUNCTIONS WHEN x IS SMALL

The limits in this section are not required in this course, but they provide the mathematically correct way to derive the derivatives of the trigonometric functions and hence are included for completeness.

$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

It may seem strange that the expression $\lim_{x\to 0}\frac{\sin x}{x}$ has a particular value. As you look at this fraction, which appears to be approaching $\frac{0}{0}$, you would assume that it is undefined or does not exist. However, consider the following table of values (where x is in radians):

x	0.1	0.01	0.001	0.0001	−0.0001	−0.001	−0.01	−0.1
$\frac{\sin x}{x}$	0.998334	0.999983	0.9999998	0.999999998	0.999999998	0.9999998	0.999983	0.998334

As $x \to 0$ from above (through positive values) it seems that $\frac{\sin x}{x} \to 1$. As $x \to 0$ from below (through negative values) it also seems that $\frac{\sin x}{x} \to 1$. You can use a spreadsheet to continue this investigation with even smaller values for x, and ultimately seem to find the result: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1}{2}$

Consider the following table of values for $\frac{1-\cos x}{x^2}$.

x	0.1	0.01	0.001	−0.001	−0.01	−0.1
$\frac{1-\cos x}{x^2}$	0.49958	0.4999958	0.49999999	0.49999999	0.4999958	0.49958

As $x \to 0$ from above (through positive values) it seems that $\frac{1-\cos x}{x^2} \to 0.5$. As $x \to 0$ from below (through negative values) it seems that $\frac{1-\cos x}{x^2} \to 0.5$. A spreadsheet investigation can confirm this for even smaller values of x, hence the result: $\lim_{x\to 0}\frac{1-\cos x}{x^2} = 0.5$

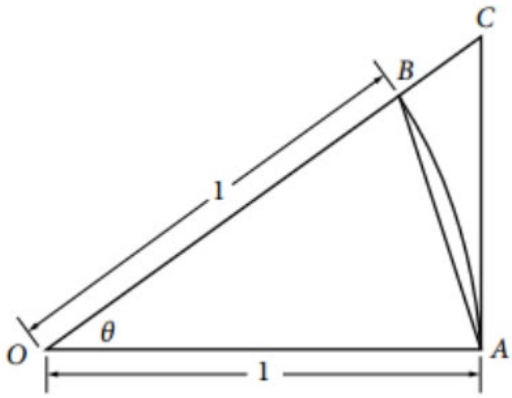
Formal proof of $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

Consider a sector OAB of a circle of unit radius.

We can draw the chord AB

Further this sector is included in a triangle OAC where AC is the tangent to the circle at A , and C is aligned with points O and B

Let the size of $\angle AOB$ be θ radians.



$$\therefore\ Area\ \Delta AOB < Area\ of\ sector\ OAB < Area\ \Delta AOC$$

$$Area\ \Delta AOB = \frac{1}{2}\times 1\times 1\times \sin\theta = \frac{1}{2}\sin\theta$$
$$Area\ of\ sector\ OAB = \frac{1}{2}\times 1^2\times\theta = \frac{\theta}{2}$$
$$Area\ \Delta AOC = \frac{1}{2}\times 1\times AC = \frac{1}{2}\times 1\times\tan\theta$$ as $\tan\theta = \frac{opposite}{adjacent} = \frac{AC}{1}$

Therefore $\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta$

Hence $\sin\theta < \theta < \tan\theta$

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

Take reciprocals $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$

Hence, multiplying both sides by $\sin\theta$, we obtain: $1 > \frac{\sin\theta}{\theta} > \cos\theta$

But $\lim_{\theta\to 0}\cos\theta = 1$ therefore the value of $\frac{\sin\theta}{\theta}$ is "squeezed" between 1 and $\cos\theta$ which value also tends towards 1 when θ tends towards 0.

Therefore $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$

Formal proof of $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\frac{1}{2}$

We consider $\frac{\sin^2\theta}{\theta^2}=\frac{1-\cos^2\theta}{\theta^2}=\frac{(1-\cos\theta)(1+\cos\theta)}{\theta^2}$

Hence $\frac{(1-\cos\theta)}{\theta^2}=\frac{1}{1+\cos\theta}\times\frac{\sin^2\theta}{\theta^2}$

Or $\frac{(1-\cos\theta)}{\theta^2}=\frac{1}{1+\cos\theta}\times\left(\frac{\sin\theta}{\theta}\right)^2$

Therefore $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{x^2}\right)=\lim_{\theta\to 0}\left[\frac{1}{1+\cos\theta}\times\left(\frac{\sin\theta}{\theta}\right)^2\right]$

But $\lim_{\theta\to 0}\left(\frac{1}{1+\cos\theta}\right)=\frac{1}{2}$ and $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1$

Therefore $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}\right)=\frac{1}{2}$

Also: $\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)=1$

$$\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)=\lim_{\theta\to 0}\left(\tan\theta\times\frac{1}{\theta}\right)=\lim_{\theta\to 0}\left(\frac{\sin\theta}{\cos\theta}\times\frac{1}{\theta}\right)=\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\times\frac{1}{\cos\theta}\right)$$

But $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1$ and $\lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right)=\frac{1}{1}=1$

Therefore $\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)=1$

These results will be used to find derivatives of trigonometric functions using first principle of differentiation (i.e. calculation of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$)

Example 1

Find: **(a)** $\lim_{x\to 0}\frac{\sin 3x}{3x}$ **(b)** $\lim_{x\to 0}\frac{\sin 4x}{2x}$ **(c)** $\lim_{x\to 0}\frac{\tan 3x}{6x}$

Solution

(a) $\lim_{x\to 0}\frac{\sin 3x}{3x}$

$=\lim_{x\to 0}\frac{\sin\theta}{\theta}$ where $\theta=3x$

$=1$

(b) $\lim_{x\to 0}\frac{\sin 4x}{2x}$

$=2\times\lim_{x\to 0}\frac{\sin 4x}{4x}$

$=2\times 1=2$

(c) $\lim_{x\to 0}\frac{\tan 3x}{6x}$

$=\frac{1}{2}\times\lim_{x\to 0}\frac{\tan 3x}{3x}$

$=\frac{1}{2}\times 1=\frac{1}{2}$