USING INDUCTION TO PROVE FIRST ORDER RECURSIVE FORMULAS

A recursive formula is when one term is defined in terms of one or more preceding terms.

A **recursive** definition of a function allows you to evaluate the function at a certain value of n using value(s) of the function at some other value(s) of n.

A **closed form** definition allows a function to be evaluated directly from the required value of n.

Example 25

A sequence $\{u_n\}$ is defined recursively as $u_1 = 2$, $u_2 = 6$, $u_n = 6u_{n-1} - 5u_{n-2}$ for $n \geq 3$.

Prove by induction the closed form definition $u_n = 5^{n-1} + 1$ for all integers $n \geq 1$.

Solution

Step 1 Prove true for the two initial cases.

$n = 1$: $u_1 = 5^{1-1} + 1 = 2$, which agrees with the recursive definition.

$n = 2$: $u_2 = 5^{2-1} + 1 = 6$, which agrees with the recursive definition.

The proposition is true for $n = 1$ and $n = 2$.

Step 2 Assume the proposition is true for all integers from 1 to k. Thus prove that it is true for $n = k + 1$.

i.e. assume:

$$\begin{aligned} u_1 &= 5^{1-1} + 1 \\ u_2 &= 5^{2-1} + 1 \\ u_3 &= 5^{3-1} + 1 \\ &\dots \\ u_{k-1} &= 5^{(k-1)-1} + 1 = 5^{k-2} + 1 \\ u_k &= 5^{k-1} + 1 \end{aligned}$$

For $n = k + 1$:

$$\begin{aligned} u_{k+1} &= 6u_{(k+1)-1} - 5u_{(k+1)-2} \qquad \text{(using the recursive definition)} \\ &= 6u_k - 5u_{k-1} \\ &= 6(5^{k-1} + 1) - 5(5^{k-2} + 1) \\ &= 6 \times 5^{k-1} + 6 - 1 \times 5^{k-1} - 5 \\ &= 5 \times 5^{k-1} + 1 \\ &= 5^k + 1 \qquad \text{which is the closed form definition.} \end{aligned}$$

Step 3 Conclusion

The proposition is true for $n = k + 1$ if it is true for $n = 1, 2, \dots k$.
It is true for $n = 1, 2$.
$\therefore$ by induction, it is true for all integers $n \geq 1$.

Example 26

(a) If $u_{r+1} = 2u_r + 1$ for all positive integer values of r, prove that $u_n + 1 = 2^{n-1}(u_1 + 1)$.

(b) Find the value of $\sum_{r=1}^{n} u_r$ if $u_1 = 1$.

Solution

(a) $r = 1$: $u_2 = 2u_1 + 1$

Consider $u_n + 1 = 2^{n-1}(u_1 + 1)$.

Step 1 When $n = 1$: LHS $= u_1 + 1$; RHS $= 2^0(u_1 + 1) = u_1 + 1 =$ LHS

Hence the result is true for $n = 1$.

When $n = 2$: LHS $= u_2 + 1 = 2u_1 + 2$; RHS $= 2^1(u_1 + 1) = 2u_1 + 2 =$ LHS

Hence the result is true for $n = 2$.

Step 2 Assume the result is true for $n = k$, given $u_{k+1} = 2u_k + 1$, i.e. assume that $u_k + 1 = 2^{k-1}(u_1 + 1)$.

Prove the result is true for $n = k + 1$, i.e. prove that $u_{k+1} + 1 = 2^k(u_1 + 1)$.

$$\begin{aligned}
\text{LHS} &= u_{k+1} + 1 \\
&= 2u_k + 1 + 1 \\
&= 2(u_k + 1) \\
&= 2\left(2^{k-1}(u_1 + 1)\right) \\
&= 2^k(u_1 + 1) \\
&= \text{RHS}
\end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

(b)
$$\begin{aligned}
u_r &= 2^{r-1}(1+1) - 1 \\
&= 2^r - 1
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}(2^r - 1) &= \sum_{r=1}^{n} 2^r - n \\
&= \frac{2(2^n - 1)}{2 - 1} - n \\
&= 2^{n+1} - n - 2
\end{aligned}$$